STABLE BIRATIONAL INVARIANTS WITH GALOIS DESCENT AND DIFFERENTIAL FORMS

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ABSTRACT. I show that the cohomology of the generic points of algebraic complex varieties becomes *stable* birational invariant, when considered 'modulo the cohomology of the generic points of the affine spaces'.

These notes are concerned with certain birational invariants of smooth algebraic varieties. All such invariants are *dominant* sheaves, cf. below; the dominant sheaves are characterized in Proposition 1.7.

Two classes of invariants are of special interest: (i) *stable*, i.e., taking the same values on a variety and on its direct product with an affine space, and (ii) constant on the projective spaces. Though the latter class is a priori wider, there are no known examples of non-stable invariants vanishing on the projective spaces. Here an attempt of comparison is made. Namely, it is shown that the corresponding adjoint functors coincide on the following types of invariants: (i) of 'level 1', cf. Proposition 2.10 and also p.15, (ii) 'related to cohomology' (or to closed differential forms).

Differential forms play a very special rôle in the story, cf. e.g. Conjecture 1.5. Moreover, all known examples of *simple* invariants (as objects of an abelian category) 'come from' differential forms: except for two invariants related to the multiplicative and the additive groups ($Y \mapsto (k(Y)^\times/k^\times)_{\mathbb{Q}}$ and $Y \mapsto k(Y)/k$, the logarithmic and the exact differentials, cf. below), they are values of the functor $\mathbb{B}^0$ from §1.3. For these reasons the differential forms are studied in detail. It is shown in Corollary 2.8 that the cohomology of the generic points of algebraic (complex) varieties becomes *stable* birational invariant, when considered 'modulo the cohomology of the generic points of the affine spaces'.

The principal new results of §3 are Propositions 3.3 and 3.7. It is shown in Proposition 3.3 that (i) the quotient $V^\bullet$ of the sheaf of algebras of closed differential forms by the ideal generated by the exact 1-forms and the logarithmic differentials is stable and (ii) $V^\bullet$ is the maximal stable quotient of the sheaf of closed differential forms. Proposition 3.7 gives a complete description of the sheaf of closed 1-forms.

Depending on what is more convenient, we shall consider our 'invariants' either as dominant sheaves, or as representations, cf. §2.5. E.g., the simplicity is more natural in the context of representations.

1. DOMINANT PRESHEAVES AND SHEAVES

Notations. From now on we fix an algebraically closed field k of characteristic zero, and denote by E a variable coefficient field of characteristic zero. Denote by Vec_E the category of E-vector spaces.

I am interested in birational invariants of (or "presheaves on") k-varieties. More precisely, let $\mathcal{S}m'_k$ be the category, whose objects are smooth k-varieties and the morphisms are smooth k-morphisms. Define the pretopology on $\mathcal{S}m'_k$ by saying that the covers are dominant morphisms. Recall, that a presheaf is a sheaf if the following diagram is an equalizer for any covering $Y \to X$:

$$(1) \qquad \mathcal{F}(X) \to \mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y \times_X Y).$$

The category of the sheaves of E-vector spaces on this site is denoted by $\mathcal{S}m_G(E)$ and $\mathcal{S}m_G := \mathcal{S}m_G(\mathbb{Q})$.

Example. For each irreducible smooth k-variety X and integer $0 \le q \le \dim X$ let $\Psi_{X,q} : Y \mapsto Z^q(k(X) \otimes_k k(Y))_{\mathbb{Q}}$ ($\mathbb{Q}$-linear combinations of irreducible subvarieties on $X \times_k Y$ of codimension q dominant over X and Y.) This is a sheaf. Set $\Psi_X := \Psi_{X,\dim X}$. The sheaves Ψ_X for all X form a system of generators of $\mathcal{S}m_G$.

Definition. 1. A presheaf $\mathcal{F}$ is $\mathbb{A}^1$-*invariant* (or *stable*) if $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times \mathbb{A}^1)$ for all X.

2. Let $\mathcal{S}$ be a collection of dominant morphisms in $\mathcal{S}m_k'$ with connected fibres. Assume that $\mathcal{S}$ is stable under base changes of its arbitrary element by itself: $\mathrm{pr}_1 : X \times_Y X \to X$ belongs to $\mathcal{S}$ if $X \to Y$ belongs to $\mathcal{S}$. A presheaf $\mathcal{F}$ is called an $\mathcal{S}$-presheaf if $\mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(X)$ for all $(X \to Y) \in \mathcal{S}$.

Denote by $\mathcal{S}m_G^{\mathcal{S}}$ the full subcategory in $\mathcal{S}m_G$ consisting of $\mathcal{S}$-sheaves. More particularly, denote by $\mathcal{I}_G$ the full subcategory in $\mathcal{S}m_G$ consisting of $\mathbb{A}^1$-invariant sheaves. Under assumptions of §1.2, $\mathcal{S}m_G^{\mathcal{S}} \subseteq \mathcal{I}_G$.

For any dominant presheaf $\mathcal{F}$ denote by $\underline{\mathcal{F}}$ its dominant sheafification.

For each smooth k-variety Y, we denote by $\overline{Y}$ a smooth compactification of Y.

1.1. Examples of $\mathbb{A}^1$-invariant presheaves.
In this section we consider some examples of dominant presheaves with values in various abelian categories. They come either from algebro-geometric constructions, or from a cohomology theory H^* (with coefficients in a commutative $\mathbb{Q}$-algebra B). As Example 5 suggests, those of these examples that are $\mathbb{A}^1$-invariant sheaves, are related. This is one of motivations for Conjecture 1.4.

An effective pure motive is a pair consisting of a smooth projective variety and a projector in the algebra of correspondences modulo numerical equivalence. Morphisms of *co(ntra)variant* pure motives are defined by correspondences modulo numerical equivalence so that they behave as action on the (co)homology.

Denote by $\mathcal{M}_k$ the category of covariant pure k-motives (and by $\mathcal{M}_k^{\mathrm{op}}$ its opposite, the category of contravariant pure k-motives). By a well-known result of U.Jannsen, these two categories are abelian and semisimple. A simple effective pure motive is called *primitive* if it is "not divisible by the Lefschetz motive", the motive $(\mathbb{P}^1, \pi)$, where π induces 0 on the 0-th and the identity on the second (co)homology.

Denote by $\overline{Y}^{\mathrm{prim}}$ the sum in the motive of $\overline{Y}$ of all its primitive submotives; CH^q is the (Chow) group of codimension q cycles modulo rational equivalence. We also use notations and identifications of §2.3.

	Invariant of a connected Y (dominant presheaf)	**Values**	**stable**
1	$K_q(Y)_{\mathbb{Q}}$ for $q \geq 0$/ its sheafification	$\mathrm{Vec}_{\mathbb{Q}}$	yes/only for $q=0$
2	$H^q(Y)$ for $q \geq 0$/ its sheafification $\underline{H}^q$	B-mod	yes/only for $q=0$
3	$\Gamma(\overline{Y}, \bigotimes_{\mathcal{O}_{\overline{Y}}}^{\bullet} \Omega^1_{\overline{Y}|k})$ / its sheafification $\Gamma(\bigotimes_F^{\bullet} \Omega^1_{F|k})$, cf. Remark on p.12	Vec_k	yes/no
4	$\Phi^p CH^q(X \times_k k(Y))_{\mathbb{Q}}$ for a smooth X and a "universal" filtration $\Phi^{\bullet}$ on the Chow groups		
	(e.g., $A(k(Y))_{\mathbb{Q}}$ for an abelian k-variety A)	$\mathrm{Vec}_{\mathbb{Q}}$	yes
5	$\overline{Y}^{\mathrm{prim}} = \bigoplus_M \overline{Y}_M^{\mathrm{prim}}$ (multiplicity-one sheaf, by Proposition 1.3)	$\mathcal{M}_k^{\mathrm{op}}$	yes
6	$Z^{\dim Y}(F \otimes_k k(Y))_{\mathbb{Q}}$	$\mathcal{S}m_G^{\mathrm{op}}$	no
7	$Z^q(Y \times_k F)_{\mathbb{Q}}$ for $q \geq 0$ / its sheafification $Z^q(F \otimes_k k(Y))_{\mathbb{Q}}$	$\mathcal{S}m_G$	only for $q=0$
	(a) composition with the evaluation functor on X, i.e., $\Psi_{X,q}$	$\mathrm{Vec}_{\mathbb{Q}}$	only for $q=0$
8	$C_{k(Y)} := \mathcal{I}\Psi_X$, cf. §1.2, (and its quotient $CH_0(\overline{Y}_F)_{\mathbb{Q}})$	$\mathcal{I}_G^{\mathrm{op}}$	yes
	(a) composition with $\mathrm{Hom}_{\mathcal{S}m_G^{\mathrm{op}}}(\underline{H}_c^q, -)$:		
	$H^q(\overline{Y})/N^1 =: \underline{H}_c^q(Y)$ for any $q \geq 0$ (subsheaf of the sheaf $\underline{H}^q$)	B-mod	yes
	(a') $k = \mathbb{C}$: the image in $\underline{H}^{2q}(-(\mathbb{C}); \mathbb{Q})(Y)$ of the maximal Hodge substructure of $H^{2q}(\overline{Y}(\mathbb{C}), \mathbb{Q})$ in F^1, cf. §3, p.12, (its vanishing is equivalent to the Hodge's conjecture)	$\mathrm{Vec}_{\mathbb{Q}}$	yes
	(b) composition with $\mathrm{Hom}_{\mathcal{S}m_G^{\mathrm{op}}}(\Omega^{\bullet}_{F|k}, -)$:		
	$\Gamma(\overline{Y}, \Omega^{\bullet}_{\overline{Y}|k})$ (subsheaf of the sheaf $\underline{H}^{\bullet}_{\mathrm{dR}|k,c}$)	Vec_k	yes

Except for $\Gamma(\overline{Y}, \bigotimes_{\mathcal{O}_{\overline{Y}}}^{\bullet} \Omega^1_{\overline{Y}|k})$, all these invariants have Galois descent property. Except for $Z^q(Y \times_k F)_{\mathbb{Q}}$ for $q > 0$, $K_q(Y)_{\mathbb{Q}}$ for $q \geq 0$ and $H^q(Y)$ for $q > 0$, all these invariants are birational. (N^1 in example 8 (a) denotes the first term of the coniveau filtration on H^*.)

Some of the above presheaves are defined using a compactification $\overline{Y}$. To show that each of such presheaves is in fact well-defined (and therefore, birationally invariant), one can use the facts that (i) any birational map is a composition of blow-ups and blow-downs with smooth centres, cf. [1], and (ii) the cohomology (resp., motive) of a blow-up is the direct sum of the cohomology of the original variety and of the Gysin image (resp., Tate twist) of the cohomology (resp., motive) of the subvariety which is blown up. Such a presheaf is $\mathbb{A}^1$-invariant, since the cohomology (resp., motive) of the product of a proper variety X with the projective line is the direct sum of the pull-back of the cohomology (resp., motive) of X and of the Gysin image (resp., Tate twist) of the cohomology (resp., motive) of $X \times \{0\} \cong X$.

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To conclude that a birational $\mathbb{A}^1$-invariant presheaf is a sheaf, one checks that it has the Galois descent property, so Proposition 1.7 can be applied.

Lemma 1.1. *For an arbitrary commutative k-group A, let $\mathcal{H}_1^A$ be the presheaf $Y \mapsto \bigoplus_{y \in Y^0}(A(k(y))/A(k))_{\mathbb{Q}}$. Then $\mathcal{H}_1^A$ is a sheaf; it is simple (=irreducible) for simple A. Let a presheaf $\mathcal{F}$ be the composition of the Picard functor $Y \mapsto \mathrm{Pic}^0(\overline{Y})$ with an additive functor on the category of abelian k-varieties, e.g. $\mathrm{Pic}^\circ_{\mathbb{Q}} : Y \mapsto \mathrm{Pic}^0(\overline{Y})_{\mathbb{Q}}$, $\underline{H}^1_c : Y \mapsto H^1(\overline{Y})$, or $\Omega^1_{|k,\mathrm{reg}} : Y \mapsto \Gamma(\overline{Y}, \Omega^1_{\overline{Y}|k})$. Then $\mathcal{F}$ is a sheaf and $\mathcal{F} = \bigoplus_A \mathcal{F}(\tilde{A}) \otimes_{\mathrm{End}(\tilde{A})} \mathcal{H}_1^{\tilde{A}}$, where A runs through the isogeny classes of simple abelian k-varieties and $\tilde{A}$ is a representative of A.*

Thus, such sheaves $\mathcal{F}$ are direct sums of copies of simple sheaves $\mathcal{H}_1^A$.

Proof. Suppose that $\mathcal{A}$ is an abelian category. Then any semisimple object $N \in \mathcal{A}$ splits *canonically* into the direct sum over the isomorphism classes M of simple objects in $\mathcal{A}$ of its M-isotypical parts N_M. Clearly, for any representative $\tilde{M}$ of the isomorphism class M the natural morphism $\mathrm{Hom}_{\mathcal{A}}(\tilde{M}, N) \otimes_{\mathrm{End}(\tilde{M})} \tilde{M} \to N_M$ by $\varphi \otimes a \mapsto \varphi(a)$. This is an isomorphism. Applying an additive functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ to the above isotypical decomposition of N, we get a canonical isomorphism $\prod_M \mathfrak{F}(\tilde{M}) \otimes_{\mathrm{End}(\tilde{M})} \mathrm{Hom}_{\mathcal{A}}(N, \tilde{M}) \xrightarrow{\sim} \mathfrak{F}(N)$, $f \otimes l \mapsto l^* f$, where the following duality is used: $\mathrm{Hom}_{\mathrm{mod}\text{-}\mathrm{End}(\tilde{M})}(\mathrm{Hom}_{\mathcal{A}}(\tilde{M}, N), \mathrm{End}(\tilde{M}))_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(N, \tilde{M})$. (It is induced by the composition pairing $\mathrm{Hom}_{\mathcal{A}}(\tilde{M}, N) \otimes \mathrm{Hom}_{\mathcal{A}}(N, \tilde{M}) \to \mathrm{End}(\tilde{M})$.)

As $\mathcal{A}$, we take either the category of abelian k-varieties with morphisms $\otimes \mathbb{Q}$, or the bigger category $\mathcal{M}_k^{\mathrm{op}}$. In the case of abelian varieties, the isomorphism classes of simple objects are the isogeny classes of simple abelian k-varieties, whereas the existence of the isotypical decomposition corresponds to the fact that for any abelian k-variety B the natural morphism $\bigoplus_A \mathrm{Hom}_{\mathrm{ab.}k\text{-var}}(\tilde{A}, B) \otimes_{\mathrm{End}(\tilde{A})} \tilde{A} \to B$, $\varphi \otimes a \mapsto \varphi(a)$, where A runs through the isogeny classes of simple abelian k-varieties and $\tilde{A}$ is a representative of A, is an isogeny.

Thus, any sheaf $\mathcal{F}$ with semisimple values in $\mathcal{A}$ also splits *canonically* into the direct sum over the isomorphism classes M of simple objects in $\mathcal{A}$ of its M-isotypical parts $\mathcal{F}_M$.

When $\mathcal{A} = \mathcal{M}_k^{\mathrm{op}}$ and $\mathcal{F}$ is the dominant sheaf $Y \mapsto \overline{Y}^{\mathrm{prim}}$, we get that $\mathcal{F}$ splits canonically into the direct sum of its M-isotypical parts $Y \mapsto \overline{Y}_M^{\mathrm{prim}}$. By Proposition 1.3, the M-isotypical part $Y \mapsto \overline{Y}_M^{\mathrm{prim}}$ is a simple sheaf.

If $B = \mathrm{Alb}(\overline{Y})$ (the Albanese variety) then $\mathrm{Hom}_{\mathrm{ab.}k\text{-var}}(B, \tilde{A}) = \tilde{A}(k(Y))/\tilde{A}(k)$, and thus, $\mathcal{F}(Y) = \mathcal{F}(B) \xrightarrow{\sim} \bigoplus_A \mathcal{F}(\tilde{A}) \otimes_{\mathrm{End}(\tilde{A})} (\tilde{A}(k(Y))/\tilde{A}(k))$. It is quite evident that $\mathcal{H}_1^{\tilde{A}}$ is a sheaf. By Proposition 1.7, in the case of abelian variety $\tilde{A}$, it suffices to check the Galois descent property, which is equivalent to the following one: for any abelian k-variety $\tilde{A}$ and any finite group H of its automorphisms such that $H_0(H, \tilde{A}) = 0$ one has $\mathcal{F}(\tilde{A})^H = 0$. Clearly, this property holds. The simplicity of the sheaf $\mathcal{H}_1^{\tilde{A}}$ follows from the fact that for any algebraically closed field extension $K|k(\tilde{A})$ and for any subvariety Z of A of positive dimension there are no proper subgroups of $\tilde{A}(K)$ containing all generic K-points of Z. (Any point of $\tilde{A}$ is a sum of generic points of $\tilde{A}$; any sum of $\dim A$ generic K-points of Z in sufficiently general position is a generic point of Z). This argument works more naturally in the context of representations, cf. §2. $\qquad \square$

Remark. For an abelian k-variety A, the sheaf $A_{\mathbb{Q}} : Y \mapsto A(k(Y))_{\mathbb{Q}}$ factors through the Albanese functor, but considered as a functor to the category of torsors over abelian k-varieties, so additive functors do not make sense and Lemma 1.1 is not applicable to this sheaf. In particular, it is not semisimple.

Propositions 3.7 and 3.5 suggest that (i) the isomorphism classes of irreducible subquotients of $\underline{H}^\bullet_c$ are the same as that of $\Omega^\bullet_{|k,\mathrm{reg}} : Y \mapsto \Gamma(\overline{Y}, \Omega^\bullet_{\overline{Y}|k})$, (ii) they can be naturally identified with the irreducible effective primitive motives, and (iii) the isomorphism classes of irreducible subquotients of $\underline{H}^\bullet$ are related to more general irreducible effective motives, such as the Tate motive $\mathbb{Q}(-1)$ in the case of $\underline{H}^1_{\mathrm{dR}/k}$.

Lemma 1.2. *Any dominant sheaf $\mathcal{F}$ with values in an abelian category with objects of finite length (e.g., in a category of finite-dimensional vector spaces) is $\mathbb{A}^1$-invariant.*

Proof. Any smooth morphism of connected smooth k-varieties is covering, so $X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta) \xrightarrow{p} X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta)/\mathfrak{S}_2$ is a cover for any X. On the other hand, it is the coequalizer of $X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta) \underset{\rightrightarrows}{\overset{id,(12)}{}}$

$X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta)$. Therefore, $\mathcal{F}(X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta)/\mathfrak{S}_2) \xrightarrow{p^*} \mathcal{F}(X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta))$ (i) is injective, (ii) factors through the $\mathfrak{S}_2$-invariants. As $(\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta)/\mathfrak{S}_2 \cong \mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta (\cong \mathbb{A}^1 \times \mathbb{G}_m)$, the source and the target of p^* are isomorphic. As they are of finite length, the inclusion p^* is an isomorphism. This implies that the involution (12) is identical on $\mathcal{F}(X \times (\mathbb{A}^1 \times \mathbb{A}^1 \smallsetminus \Delta))$, so in the exact sequence, defining the sheaf condition for the cover $X \times \mathbb{A}^1 \longrightarrow X$, $0 \to \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1) \rightrightarrows \mathcal{F}(X \times \mathbb{A}^1 \times \mathbb{A}^1)$ the double arrow consists of equal morphisms, i.e. $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X \times \mathbb{A}^1)$. $\qquad\square$

1.2. **Properties of $\mathcal{S}\mathrm{m}_G^{\mathcal{S}}$.** Clearly, a subsheaf of an $\mathcal{S}$-sheaf is an $\mathcal{S}$-sheaf: if $\mathcal{G}$ is a subsheaf of an $\mathcal{S}$-sheaf $\mathcal{F}$ then for any $(Y \to X) \in \mathcal{S}$ the parallel arrows in the upper line in the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) & \rightrightarrows & \mathcal{F}(Y \times_X Y) \\
\cup & & \cup & & \cup \\
\mathcal{G}(X) & \longrightarrow & \mathcal{G}(Y) & \rightrightarrows & \mathcal{G}(Y \times_X Y)
\end{array}
$$

coincide, so the parallel arrows in the lower line also coincide, i.e. $\mathcal{G}$ is an $\mathcal{S}$-sheaf.

Assume that there are generically non-finite morphisms in $\mathcal{S}$ *with arbitrary targets*. Thus as before, $\mathcal{I}_G$ is a particular case of $\mathcal{S}\mathrm{m}_G^{\mathcal{S}}$. Moreover, as restriction of any morphism $X \xrightarrow{f} Y$ to an open dense subset U of X factors through $U \xrightarrow{(f,\phi)} Y \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}_Y} Y$, one has $\mathcal{S}\mathrm{m}_G^{\mathcal{S}} \subseteq \mathcal{I}_G$.

1. The categories $\mathcal{S}\mathrm{m}_G^{\mathcal{S}}$ and $\mathcal{S}\mathrm{m}_G$ are abelian, complete, cocomplete and have enough injectives. (This is standard.)

2. The section functors $\mathrm{Hom}_{\mathcal{S}\mathrm{m}_G}(\Psi_Y, -) : \mathcal{F} \mapsto \mathcal{F}(Y)$ are exact on $\mathcal{S}\mathrm{m}_G^{\mathcal{S}}$ for all smooth k-varieties Y. As a consequence, quotients of $\mathcal{S}$-sheaves by their subsheaves coincide with their quotients as presheaves: if $\mathcal{F} \in \mathcal{S}\mathrm{m}_G^{\mathcal{S}}$ and $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ then $(\mathcal{F}/\mathcal{G})(Y) = \mathcal{F}(Y)/\mathcal{G}(Y)$.

3. A sheaf is an $\mathcal{S}$-sheaf if and only if all its irreducible subquotients are $\mathcal{S}$-sheaves.

[Proof of the "only if" part. As it was shown above, a subsheaf $\mathcal{G}$ of $\mathcal{F} \in \mathcal{S}\mathrm{m}_G^{\mathcal{S}}$ is an $\mathcal{S}$-sheaf. By property 2, $(\mathcal{F}/\mathcal{G})(Y) = \mathcal{F}(Y)/\mathcal{G}(Y)$, which implies that the quotient $\mathcal{F}/\mathcal{G}$ is also an $\mathcal{S}$-sheaf. The "if" part is shown in Proposition 2.2 (in the language of representations); cf. also Theorem 2.11.]

4. The inclusion $\mathcal{I}_G \hookrightarrow \mathcal{S}\mathrm{m}_G$ admits a left adjoint $\mathcal{I}$ and a right adjoint.

Examples of calculation of these adjoint functors are given in Propositions 3.1 and 3.3.

5. The sheaves $C_{k(X)} := \mathcal{I}\Psi_X$ form a system of projective generators of $\mathcal{I}_G$. [This follows from 2 and 4.] (*Remark.* There are no projective objects in $\mathcal{S}\mathrm{m}_G$.)

1.3. **Irreducible objects of $\mathcal{I}_G$.** *Examples.* Let M be a simple effective primitive pure covariant motive. Then

$$\mathbb{B}^0(M) : Y \mapsto \mathrm{Hom}_{\{\text{pure } k\text{-motives}\}}(\overline{Y}, M)$$

is a well-defined sheaf of finite-dimensional $\mathbb{Q}$-vector spaces ([7]).

A particular case of this example is the sheaf $\mathcal{H}_1^A$, corresponding to the motive "$H_1(A)$" for any simple abelian k-variety A.

Proposition 1.3 ([7]). $\mathbb{B}^0$ *gives rise to a fully faithful functor $\mathbb{B}^\bullet$:*

{pure k-motives} $\longrightarrow$ {semisimple sheaves of finite length of finite-dimensional graded $\mathbb{Q}$-vector spaces}.

Conjecture 1.4 ([7]). *This is an equivalence of categories.* (In other words, any irreducible sheaf of finite-dimensional $\mathbb{Q}$-vector spaces is isomorphic to $\mathbb{B}^0(M)$ for a primitive irreducible effective pure motive M.)

This can be complemented by the following conjecture, which I consider as one of the principal problems on $\mathbb{A}^1$-invariant sheaves.

Conjecture 1.5 ([8]). *Any simple $\mathbb{A}^1$-invariant sheaf can be embedded into the sheaf $\underline{\Omega}^\bullet_{|k} : Y \mapsto \Omega^\bullet_{k(Y)|k}$.*

This conjecture is rather strong: it implies the Bloch's conjecture:

"Corollary" 1.6 ([8]). *Suppose that a rational map $f : Y \dashrightarrow X$ of smooth proper k-varieties induces an injection $\Gamma(X, \Omega^\bullet_{X|k}) \hookrightarrow \Gamma(Y, \Omega^\bullet_{Y|k})$.[1] Then f induces a surjection $CH_0(Y) \to CH_0(X)$.*

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[1]*Example.* Let $r \geq 1$ be an integer and X be a smooth proper k-variety with $\Gamma(X, \Omega^j_{X|k}) = 0$ for all $r < j \leq \dim X$. Let Y be a sufficiently general r-dimensional plane section of a smooth projective variety X' birational to X. Then, as all considered invariants are birational, the inclusion $Y \hookrightarrow X'$ induces an injection $\Gamma(X, \Omega^\bullet_{X|k}) \hookrightarrow \Gamma(Y, \Omega^\bullet_{Y|k})$.

If $\Gamma(X, \Omega^{\geq 2}_{X|k}) = 0$ *then the Albanese map induces an isomorphism* $CH_0(X)^0 \xrightarrow{\sim} \mathrm{Alb}(X)(k)$, *where* $CH_0(X)^0$ *is the Chow group of 0-cycles of degree 0 and* $\mathrm{Alb}(X)$ *is the Albanese variety of* X. *(The converse, due to Mumford, is well-known.)* *In that case* $C_{k(X)} = CH_0(X_F)_{\mathbb{Q}}$.

Proof. Let C be the cokernel of $\alpha : CH_0(Y_F)_{\mathbb{Q}} \to CH_0(X_F)_{\mathbb{Q}}$. Then the kernel of the homomorphism $\alpha^* : \mathrm{Hom}_G(CH_0(X_F), \Omega^{\bullet}_{F|k}) \to \mathrm{Hom}_G(CH_0(Y_F), \Omega^{\bullet}_{F|k})$ is $\mathrm{Hom}_G(C, \Omega^{\bullet}_{F|k})$. By Proposition 3.1, the homomorphism α^* coincides with the pull-back under $f^* : \Gamma(X, \Omega^{\bullet}_{X|k}) \to \Gamma(X, \Omega^{\bullet}_{Y|k})$. As the latter is injective, we conclude that $\mathrm{Hom}_G(C, \Omega^{\bullet}_{F|k}) = 0$. If $C \neq 0$ then it is cyclic, and thus, admits an simple quotient, and therefore, a non-zero morphism to $\Omega^{\bullet}_{F|k}$. This contradiction implies that $C = 0$.

As the objects $\mathbb{Q}$ and $\mathrm{Alb}X(F)_{\mathbb{Q}}$ of $\mathcal{I}_G$ are projective ([7, §6.2]), the natural surjections $\deg : C_{k(X)} \to \mathbb{Q}$ and $\mathrm{Alb}_F : \ker \deg \to \mathrm{Alb}X(F)_{\mathbb{Q}}$ are split, so the cyclic G-module $C_{k(X)}$ is isomorphic to a direct sum of type $\mathbb{Q} \oplus \mathrm{Alb}X(F)_{\mathbb{Q}} \oplus \ker \mathrm{Alb}_F$. Thus, $\mathrm{Hom}_G(C_{k(X)}, \Omega^{\bullet}_{F|k}) \cong \mathrm{Hom}_G(\mathbb{Q} \oplus \mathrm{Alb}X(F), \Omega^{\bullet}_{F|k}) \oplus \mathrm{Hom}_G(\ker \mathrm{Alb}_F, \Omega^{\bullet}_{F|k})$. By Proposition 3.1, $\mathrm{Hom}_G(C_{k(X)}, \Omega^{\bullet}_{F|k}) = \Gamma(X, \Omega^{\bullet}_{X|k})$ and $\mathrm{Hom}_G(\mathbb{Q} \oplus \mathrm{Alb}X(F), \Omega^{\bullet}_{F|k}) = \Gamma(X, \Omega^{\leq 1}_{X|k})$. If $\Gamma(X, \Omega^{\geq 2}_{X|k}) = 0$ this means that $\mathrm{Hom}_G(C_{k(X)}, \Omega^{\bullet}_{F|k}) = \mathrm{Hom}_G(\mathbb{Q} \oplus \mathrm{Alb}X(F), \Omega^{\bullet}_{F|k})$. Therefore, the G-module $\ker \mathrm{Alb}_F$ should be zero, as otherwise it is cyclic, thus admits a non-zero simple quotient, and (by Conjecture 1.5) a non-zero morphism to $\Omega^{\bullet}_{F|k}$. It remains to take the G-invariants of $\ker \deg \xrightarrow{\sim} CH_0(X_F)^0_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Alb}X(F)_{\mathbb{Q}}$; the torsion is controlled by Roitman's theorem. $\square$

Also this would imply that any irreducible $\mathbb{A}^1$-invariant sheaf is a sheaf of finite-dimensional vector spaces.

Example. Let $\mathcal{F}$ be a simple $\mathbb{A}^1$-invariant sheaf and suppose that it is of level 1, i.e. it is non-constant and $\mathcal{F}(Y) \neq 0$ for a curve Y, cf. also p.15. Then, by [7, Corollary 6.22], $\mathcal{F} \cong \mathcal{H}^A_1$ for a simple abelian variety A. Now any non-zero $\eta \in \Gamma(A, \Omega^1_{A|k})$ gives an embedding $\mathcal{F} \hookrightarrow \Omega^1_{|k}$ by $[x : \mathcal{O}(U) \to k(Y)] \mapsto x(\eta) \in \Omega^1_{Y|k}(Y)$ ($U \subset A$ is an affine open subset).

Proposition 1.7. *A dominant presheaf* $\mathcal{F}$ *is a sheaf if and only if the following three conditions hold:* (i) *the sequence* $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1) \rightrightarrows \mathcal{F}(X \times \mathbb{A}^2)$ *is exact for any smooth k-variety X,*[2] (ii) $\mathcal{F}$ *is birationally invariant,* (iii) *it has the Galois descent property, i.e.* $\mathcal{F}(X) = \mathcal{F}(Y)^{\mathrm{Aut}(Y|X)}$ *for any Galois covering* $Y \to X$.

Proof. The conditions (i)–(iii) are particular cases of the equalizer diagram (1) for coverings by (i) projections $X \times \mathbb{A}^s \to X$, (ii) open dense $U \subset X$, (iii) étale Galois covers $Y \to X$, respectively. Galois descent property for any sheaf is clear, since étale morphisms with dense images are covering and $U \times_X U = \coprod_{g \in \mathrm{Aut}(Y|X)} U_g$ for a Zariski open $\mathrm{Aut}(Y|X)$-invariant $U \subset Y$, where $U_g \cong U$ is the image of the embedding $(id_U, g) : U \hookrightarrow U \times_X U$.

Conversely, it is clear that any Galois-separable presheaf $\mathcal{F}$ satisfying (i) and (ii) is separable: if $Y \to X$ is a cover, i.e. a smooth dominant morphism, then for any sufficiently general dominant map $\varphi : Y \dashrightarrow \mathbb{A}^\delta$ (where $\delta = \dim Y - \dim X$) we can choose a dominant étale morphism $\widetilde{Y} \to Y$ so that the composition $\widetilde{Y} \to Y \dashrightarrow X \times \mathbb{A}^\delta$ is Galois with the group denoted by H, and therefore, the composition

$$
\begin{array}{ccccc}
\mathcal{F}(X) \xrightarrow{(i)} \mathcal{F}(X \times \mathbb{A}^1) \xrightarrow{(i)} \cdots \xrightarrow{(i)} & \mathcal{F}(X \times \mathbb{A}^\delta) & \longrightarrow & \mathcal{F}(Y) \\
& \downarrow \text{injective} & & \downarrow \\
& \mathcal{F}(\widetilde{Y})^H & \hookrightarrow & \mathcal{F}(\widetilde{Y})
\end{array}
$$

is injective. Then in the commutative diagram

$$
(2) \quad
\begin{array}{ccccc}
\mathcal{F}(X) & \to & \mathcal{F}(\widetilde{Y}) & \rightrightarrows & \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y}) \\
\| & & \uparrow & & \uparrow \\
\mathcal{F}(X) & \to & \mathcal{F}(Y) & \rightrightarrows & \mathcal{F}(Y \times_X Y) \\
\| & & \uparrow & & \uparrow \\
\mathcal{F}(X) & \to & \mathcal{F}(X \times \mathbb{A}^\delta) & \rightrightarrows & \mathcal{F}(X \times \mathbb{A}^\delta \times \mathbb{A}^\delta)
\end{array}
$$

all arrows are injective, so it suffices to show the exactness of the upper row.

---

[2]E.g., any $\mathbb{A}^1$-invariant presheaf $\mathcal{F}$ satisfies the condition (i).

Let f be an element of $\mathcal{F}(\widetilde{Y})$. The image of f in $\mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})$ under the projection to the first factor is fixed by $\{1\} \times H$; the image of f in $\mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})$ under the projection to the second factor is fixed by $H \times \{1\}$. Now if f is an element of the equalizer of $\mathcal{F}(\widetilde{Y}) \rightrightarrows \mathcal{F}(\widetilde{Y} \times_X \widetilde{Y})$ then the two images coincide, so they are fixed by the group $H \times H$. The injectivity of both parallel arrows in the upper row of the diagram (2) implies that $f \in \mathcal{F}(\widetilde{Y})^H$. By (iii) and the injectivity of the vertical arrow, f comes from the equalizer of the bottom row of the diagram (2). Finally, the bottom row of the diagram (2) is exact by Lemma 1.8, and thus, f comes from $\mathcal{F}(X)$. $\qquad\square$

Lemma 1.8. *Let $\mathcal{V}$ be a category of schemes such that for any $X \in \mathcal{V}$: (i) the projection $X \times \mathbb{A}^1 \to X$ is a morphism in $\mathcal{V}$, (ii) any linear automorphism of any affine space $\mathbb{A}$ induces an automorphism of $X \times \mathbb{A}$ in $\mathcal{V}$. Let $\mathcal{F}$ be a presheaf on this category such that the sequence $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1) \rightrightarrows \mathcal{F}(X \times \mathbb{A}^2)$ is exact for any $X \in \mathcal{V}$. Then the sequence $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^s) \rightrightarrows \mathcal{F}(X \times \mathbb{A}^{2s})$ is exact for any $X \in \mathcal{V}$ and any $s \geq 1$.*

Proof. We proceed by induction on s, the case $s = 1$ being trivial.

Denote by $\mathrm{pr}_1, \mathrm{pr}_2 : X \times \mathbb{A}^{2s} \rightrightarrows X \times \mathbb{A}^s$ the two projections.

For any $f \in \mathcal{F}(X \times \mathbb{A}^s)$ the element $\mathrm{pr}_1^* f$ is fixed by $\Phi^* \in \mathrm{End}\,\mathcal{F}(X \times \mathbb{A}^s \times \mathbb{A}^s)$ for any linear automorphism $\Phi(u, v) = (u, \varphi(u, v))$ of $\mathbb{A}^s \times \mathbb{A}^s$. Similarly, $\mathrm{pr}_2^* f$ is fixed by $\Psi^* \in \mathrm{End}\,\mathcal{F}(X \times \mathbb{A}^s \times \mathbb{A}^s)$ for any linear automorphism $\Psi(u, v) = (\psi(u, v), v)$.

Let now $f \in \mathcal{F}(X \times \mathbb{A}^s)$ be in the equalizer of pr_1^* and pr_2^*. Then $\mathrm{pr}_1^* f = \mathrm{pr}_2^* f$ is fixed by the group, generated by Φ^* and Ψ^* as above. Clearly, such automorphisms Φ and Ψ generate the group consisting of all linear automorphisms α. Then $\mathrm{pr}_1^* f = \alpha^* \mathrm{pr}_1^* f$.

Applying the induction assumption in the case where α is identical on one of the first s coordinates and interchanges i-th and $(s+i)$-th for other $1 \leq i \leq s$, we get that f belongs to the image of $\mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X \times \mathbb{A}^s)$ under morphism induced by the projection $\mathbb{A}^s \to \mathbb{A}^1$ to one of the copies of $\mathbb{A}^1$. Then the case $s = 1$ implies that f comes from $\mathcal{F}(X)$. $\qquad\square$

EXAMPLES. 1. A stable birationally invariant dominant presheaf with the Galois descent is a sheaf.

2. Example of a birationally invariant presheaf $\mathcal{F}$ with the Galois descent property which is not a sheaf. Let $\mathcal{G}$ be a dominant sheaf and $I \subsetneq \{0, 1, 2, \dots\}$ be a non-empty (finite or infinite) interval. Assume that $\mathcal{G}(X) \neq 0$ for some X with $\dim X \notin I$. Then the presheaf $\mathcal{F} : U \mapsto \begin{cases} \mathcal{G}(U) & \text{if } \dim U \in I \\ 0 & \text{if } \dim U \notin I \end{cases}$ (with the restriction maps of $\mathcal{G}$, whenever possible, otherwise zero) is birationally invariant and has the Galois descent property, but it is not a sheaf. The sheafification of $\mathcal{F}$ is $\mathcal{G}$ if I is infinite and 0 otherwise.

Now, what are the projective generators of $\mathcal{I}_G$ from §1.2, Property 5?

Conjecture 1.9. *For any smooth proper k-variety X, the sheaf $C_{k(X)}$ coincides with $Y \mapsto CH_0(X_{k(Y)})_{\mathbb{Q}}$.*

Remarks. 1. This is known, e.g., if X is a curve, cf. [7, Cor.6.21] and Proposition 2.10 for a stronger statement. Conjecture 1.9 would imply that $\mathcal{I}_G$ is a tensor category under the operation $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{I}(\mathcal{F} \otimes_{\mathcal{S}\mathrm{m}_G} \mathcal{G}) =: \mathcal{F} \otimes_{\mathcal{I}} \mathcal{G}$, where $\otimes_{\mathcal{S}\mathrm{m}_G}$ denotes the sheafification of the tensor product presheaf. Moreover, the "Künneth formula" holds: $C_{k(X)} \otimes_{\mathcal{I}} C_{k(Y)} = C_{k(X \times_k Y)}$.

2. It is shown in [7, Proposition 6.17] that, roughly speaking, $C_{k(X)}$ is the quotient of generic 0-cycles on X by those divisors of rational functions on generic curves on X which are generic, and thus, Conjecture 1.9 should be considered as a moving lemma.

3. Conjecture 1.9 and the motivic conjectures imply conjectures 1.4 and 1.5.

2. ALTERNATIVE DESCRIPTIONS OF $\mathbb{A}^1$-INVARIANT SHEAVES

Now I want to introduce the language of representations and to use it to explain some results and conjectures of §1, especially Conjecture 1.9.

2.1. Smooth representations and non-degenerate modules over algebras of measures.
For any totally disconnected Hausdorff group[3] H an H-set (group, etc.) is called *smooth* if the stabilizers are open.

---

[3]cf. [4, Appendix A]

Any smooth representation W of H over E can be considered as a module over the associative algebra $\mathbb{D}_E(H) := \varprojlim_U E[H/U]$ of the "oscillating" measures on H (for which all open subgroups and their translates are measurable): $\mathbb{D}_E(H) \times W \to W$ is defined by $(\alpha, w) \mapsto \beta w$ for any $\beta \in E[H]$ with the same image $E[H]$ as α, where $w \in W^U$ for some open subgroup U of H.

Passing to the inverse limit, we get the algebra structure on $\mathbb{D}_E(H)$ from $\mathbb{D}_E(H) \times E[H/U] \to E[H/U]$.

If the annihilator of $W \in \mathcal{S}m_H(E)$ in $\mathbb{D}_E(H)$ vanishes then the restriction of W to any compact subgroup U contains each smooth irreducible representation of U. (Otherwise, if W does not contain a smooth irreducible representation ρ of U then the natural projector in $\mathbb{D}_E(H)$ to the ρ-isotypical part would annihilate W.)

2.2. A representation theoretic setting for ($\mathbb{A}^1$-invariant) sheaves.
In this section, for a group H as in §2.1 and a collection S of pairs of its subgroups, we study the category $\mathcal{S}m_H^S(E)$ of smooth E-representations W of H, satisfying $W^{U_1} = W^{U_2}$ for all $(U_1, U_2) \in S$.

Theorem 2.3 explains the consistence of this notation with that of §1.

Collections S and S' are called *equivalent* if they define the same subcategory of $\mathcal{S}m_H := \mathcal{S}m_H^\emptyset$.

For any subgroup $U \subset H$ the functor $H^0(U, -)$ on the category of smooth H-sets (or modules, etc.) coincides with $\varinjlim H^0(V, -)$, where the limit is taken over the open subgroups V of H containing U. Therefore, one can assume that the subgroups U_1, U_2 are intersections of open ones, and in particular, that they are closed. Further, as $W^{U_1} \cap W^{U_2} = W^{\langle U_1, U_2 \rangle}$ for any H-module W and $U_1, U_2 \subseteq \langle U_1, U_2 \rangle$, one can assume that the pairs $(U_1, U_2) \in S$ are ordered: $U_1 \subset U_2$.

Lemma 2.1. *Assume that for any pair $(U_1 \subset U_2) \in S$ the functor $H^0(U_1, -)$ is exact on $\mathcal{S}m_H$. Then the category $\mathcal{S}m_H^S(E)$ is stable under passing to the subquotients in $\mathcal{S}m_H(E)$, and in particular, it is abelian. The inclusion functor $\mathcal{S}m_H^S(E) \hookrightarrow \mathcal{S}m_H(E)$ admits a left adjoint[4] $W \longmapsto \mathcal{I}_S W$.*

Proof. If a sequence $0 \to W_1 \to W \to W_2 \to 0$ in $\mathcal{S}m_H$ is exact then the sequences $0 \to W_1^{U_1} \to W^{U_1} \to W_2^{U_1} \to 0$ and $0 \to W_1^{U_2} \to W^{U_2} \to W_2^{U_2}$ are also exact. If $W \in \mathcal{S}m_H^S$, i.e. $W^{U_1} = W^{U_2}$, then $W_1^{U_1} = W_1 \cap W^{U_1} = W_1 \cap W^{U_2} = W_1^{U_2}$ and $W^{U_2} \to W_2^{U_2}$ is surjective (since $W^{U_2} = W^{U_1} \to W_2^{U_1}$ is surjective and factors through $W_2^{U_2} \subseteq W_2^{U_1}$). This means that $\mathcal{S}m_H^S$ is stable under taking subquotients in $\mathcal{S}m_H$.

The existence of the functor $\mathcal{I}_S$ can be deduced from the special adjoint functor theorem, cf. [5, §5.8]. However, we construct it "explicitly", which enables us to relate the generators of the category $\mathcal{I}_G$ to the Chow groups of 0-cycles.

Let $W' \in \mathcal{S}m_H^S$. Any H-homomorphism $W \xrightarrow{\alpha} W'$ factors through the object $\alpha(W)$ of $\mathcal{S}m_H^S$. We may, therefore, assume that α is surjective. Let $(U_1 \subset U_2) \in S$. As the functor $H^0(U_1, -)$ is exact on $\mathcal{S}m_H$, the morphism α induces a surjection $W^{U_1} \longrightarrow (W')^{U_1}$. As $(W')^{U_2} = (W')^{U_1}$, the subgroup U_2 acts on $(W')^{U_1}$ trivially, and therefore, the subrepresentation $W_{U_1 \subset U_2} = \langle \sigma w - w \mid \sigma \in U_2, \ w \in W^{U_1} \rangle_H$ of H is contained in the kernel of α. It follows that α factors through $\mathcal{I}_S W := W / \sum_{(U_1 \subset U_2) \in S} W_{U_1 \subset U_2}$.

The representation $\mathcal{I}_S W$ of H is smooth, so the map $W^{U_1} \longrightarrow (\mathcal{I}_S W)^{U_1}$, induced by the projection, is surjective, and therefore, any element $\overline{w} \in (\mathcal{I}_S W)^{U_1}$ can be lifted to an element $w \in W^{U_1}$. Then $\sigma \overline{w} - \overline{w}$ coincides with the projection of the element $\sigma w - w$ for any $\sigma \in U_2$. Notice that $\sigma w - w \in W_{U_1 \subset U_2}$, so its projection is zero, and therefore, $\sigma \overline{w} = \overline{w}$ for any $\sigma \in U_2$. As $(\mathcal{I}_S W)^{U_2} \subseteq (\mathcal{I}_S W)^{U_1}$, this means that $(\mathcal{I}_S W)^{U_2} = (\mathcal{I}_S W)^{U_1}$, and thus, $\mathcal{I}_S W \in \mathcal{S}m_H^S$.

One has $\mathrm{Hom}_{\mathcal{S}m_H^S}(\mathcal{I}_S W, W') = \mathrm{Hom}_{\mathcal{S}m_H}(W, W')$ for any $W \in \mathcal{S}m_H$ and $W' \in \mathcal{S}m_H^S$, i.e. the functor $\mathcal{I}_S$ is left adjoint to the inclusion functor $\mathcal{S}m_H^S \hookrightarrow \mathcal{S}m_H$. $\square$

Remark. The functor $\mathcal{I}_S$ generalizes the coinvariants, since $\mathcal{I}_S = H_0(H, -)$ if $S = \{(\{1\} \subset H)\}$.

EXAMPLES. 1. The functor $H^0(U_1, -)$ is exact on $\mathcal{S}m_H$ if, e.g., the subgroup U_1 is compact.

---

[4]The diagrams
$$\begin{array}{ccc} \mathcal{S}m_H(E) & \xrightarrow{\mathcal{I}_S} & \mathcal{S}m_H^S(E) \\ {\scriptstyle \otimes_E E'} \downarrow & & \downarrow {\scriptstyle \otimes_E E'} \\ \mathcal{S}m_H(E') & \xrightarrow{\mathcal{I}_S} & \mathcal{S}m_H^S(E') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{S}m_H(E) & \xrightarrow{\mathcal{I}_S} & \mathcal{S}m_H^S(E) \\ {\scriptstyle \text{for}} \uparrow & & \uparrow {\scriptstyle \text{for}} \\ \mathcal{S}m_H(E') & \xrightarrow{\mathcal{I}_S} & \mathcal{S}m_H^S(E') \end{array}$$
are commutative for any field extension $E'|E$, so omitting E from the notation does not lead to a confusion.

2. Suppose that H is the automorphism group of an algebraically closed field extension $F|k$ of countable transcendence degree and U_1 is the subgroup of automorphisms of F over a fixed subextension of k in F of infinite transcendence degree. Though U_1 need not be compact, the functor $H^0(U_1, -)$ is exact on $\mathcal{S}m_H$.

Proposition 2.2. *Let H be a totally disconnected group and S be such a collection of pairs of its subgroups $(U_1 \subset U_2)$ that*

(1) *for any pair $(U_1 \subset U_2) \in S$ there exists an element $\sigma \in U_2$ such that* (i) *$(U_1 \cap \sigma U_1 \sigma^{-1} \subset U_1) \in S$;* (ii) *$U_1$ and $\sigma U_1 \sigma^{-1}$ generate U_2, at least topologically.*

(2) *there exists an equivalent collection of pairs of its subgroups $(U_1 \subset U_2)$, where all U_1 are compact.*

Then an object of $\mathcal{S}m_H(E)$ belongs to $\mathcal{S}m_H^S(E)$ if and only if all its irreducible subquotients are in $\mathcal{S}m_H^S(E)$. In particular, $\mathcal{S}m_H^S(E)$ is a Serre subcategory of $\mathcal{S}m_H(E)$.

Proof. Suppose that $W \notin \mathcal{S}m_H^S$, whereas all its irreducible subquotients are in $\mathcal{S}m_H^S$. Then $W^{U_1} \neq W^{U_2}$ for some pair $(U_1 \subset U_2) \in S$, that is there exist a vector $v \in W^{U_1} \smallsetminus W^{U_2}$. Choose an element $\sigma \in U_2$ as in condition (1) of the statement for the pair $(U_1 \subset U_2) \in S$. Then $\sigma v - v =: u \neq 0$, since U_1 and σ generate a dense subgroup in U_2.

One may replace W by its quotient by a maximal subrepresentation not containing u. Then the subrepresentation $\langle u \rangle$, generated by u, becomes irreducible, and thus, an object of $\mathcal{S}m_H^S$.

By definition, $u \in W^{U_1} + W^{\sigma U_1 \sigma^{-1}} \subseteq W^{U_1 \cap \sigma U_1 \sigma^{-1}}$. As $\langle u \rangle \in \mathcal{S}m_H^S$ and $(U_1 \cap \sigma U_1 \sigma^{-1} \subset U_1) \in S$, we conclude that $u \in W^{U_1}$. This implies that $\sigma v \in W^{U_1}$. On the other hand, $\sigma v \in W^{\sigma U_1 \sigma^{-1}}$, so $\sigma v \in W^{U_1} \cap W^{\sigma U_1 \sigma^{-1}}$. The latter vector space coincides with W^{U_2}, and thus, $v \in W^{U_2}$, contradicting our assumptions.

The converse follows from Lemma 2.1. $\qquad\square$

2.3. **More notations and compatibility of notations of §2.2 and §1: the sheafification and smooth representations.** From now on we fix the following notations: $F|k$ is an algebraically closed field extension of countably infinite transcendence degree, and $G = G_{F|k}$ is the automorphism group of the extension $F|k$.

Consider connected smooth k-varieties U endowed with a generic F-point, i.e., with a k-field embedding $k(U) \overset{/k}{\hookrightarrow} F$. For any presheaf $\mathcal{F}$ on $\mathcal{S}m_k'$ we can form the direct limit $\mathcal{F}(F) := \varinjlim \mathcal{F}(U)$ over such U. The group $G = \mathrm{Aut}(F|k)$ acts naturally on $\mathcal{F}(F)$.

Theorem 2.3 ([4]).
- $\mathcal{F} \mapsto \mathcal{F}(F)$ *gives an equivalence of the categories $\mathcal{S}m_G^{\mathcal{S}}(E)$ (of §1) and $\mathcal{S}m_G^{S}(E)$ (of §2.2), where S is the collection of pairs $G_{F|k(X)} \subseteq G_{F|k(Y)}$ for all morphisms $(X \to Y) \in \mathcal{S}$.*
- *For any presheaf $\mathcal{F}$, the sheaf corresponding to $\mathcal{F}(F)$ is the sheafification of $\mathcal{F}$.*

2.4. **An example: birational invariants constant on the projective spaces.** Let S consist of a single pair $K \subset G$ such that K is a 'maximal' compact subgroup, i.e., any compact subgroup is conjugate to a subgroup of K. Then S is equivalent to the collection consisting of a single pair $K' \subset G$, where K' is the pointwise stabilizer of some transcendence base of $F|k$, and also to the collection S' of pairs $U \subset G$ such that U is the pointwise stabilizer of a finite subset of a fixed transcendence base of $F|k$. The collection S' satisfies the assumptions of Proposition 2.2.

Lemma 2.4. *Let $E'|E$ be an extension of fields, H be a group and (ρ, W_2) be an irreducible E'-representation of H. Let W_1 be an E-representation of H, absolutely irreducible even in restriction to $\ker \rho$.[5] Then the E'-representation $W_1 \otimes_E W_2$ of H is irreducible.*

Proof. Let $\xi \in W_1 \otimes_E W_2$ be a non-zero vector. It suffices to check that the $E'[H]$-span of ξ contains $W_1 \otimes v$ for any $v \in W_2$. Any non-zero $E[\ker \rho]$-submodule in W_1^m is isomorphic to $W_1^{m'}$ for some $1 \leq m' \leq m$, and therefore, the $E[\ker \rho]$-submodule in $W_1 \otimes_E W_2$ spanned by ξ (which is in fact a submodule in $\oplus_{i=1}^m W_1 \otimes v_i \cong W_1^m$ for some $m \geq 1$ and E-linearly independent $v_1, \ldots, v_m$) contains a $E[\ker \rho]$-submodule W_1' isomorphic to W_1. As the endomorphisms of the $E[\ker \rho]$-module W_1 are scalar, there exists a non-zero m-tuple $(a_1, \ldots, a_m) \in E^n$ such that $W_1' = \{a_1 w \otimes v_1 + \cdots + a_m w \otimes v_m \mid w \in W_1\}$. In other words, $W_1' = W_1 \otimes v'$, where $v' := a_1 v_1 + \cdots + a_m v_m$ is a non-zero vector in W_2.

---

[5]i.e., irreducible and with $\mathrm{End}_{E[\ker \rho]}(W_1) = E$: otherwise, if $\mathrm{End}_{E[\ker \rho]}(W_1) \neq E$ and $E'|E$ is a non-trivial field extension in the division E-algebra $\mathrm{End}_{E[\ker \rho]}(W_1)$ then the action of E' on W_1 gives a non-injective surjection of E'-representations $W_1 \otimes_E E' \longrightarrow W_1$.

As any vector v of W_2 is an E'-linear combination of several elements in the H-orbit of v', we may assume that $v = hv'$ for some $h \in H$. Then $u \otimes v = h(h^{-1}u \otimes v')$ for any $u \in W_1$. $\qquad\square$

Lemma 2.5. *Let $E'|E$ be an extension of fields, H be a group and (ρ, W_2) be an irreducible E'-representation of H. Let W_1 be an E-representation of H such that* (i) *the sum Σ of all proper E-subrepresentations of* $\ker \rho$ *in W_1 is proper*[6] *and* (ii) *W_1/Σ is absolutely irreducible in restriction to $\ker \rho$ and its restriction to the pointwise stabilizer Ξ of Σ in $\ker \rho$ is non-trivial. Then any proper E'-subrepresentation of H in $W_1 \otimes_E W_2$ is contained in $\Sigma \otimes_E W_2$.*

Proof. Let $\xi \in W_1 \otimes_E W_2$ be a vector, which is not in $\Sigma \otimes_E W_2$. It suffices to check that the $E'[H]$-span V of ξ contains $W_1 \otimes v$ for some non-zero $v \in W_2$, as then V coincides with $W_1 \otimes_E W_2$: any vector of W_2 is an E'-linear combination of several elements in the H-orbit of v and $W_1 \otimes hv = h(W_1 \otimes v)$ for any $h \in H$.

It follows from Lemma 2.4 that V is surjective over $(W_1/\Sigma) \otimes_E W_2$. In particular, V contains an element of type $\sum_{i=1}^{m} a_i \otimes b_i$ for some $a_1 \in W_1 \smallsetminus \Sigma$, whose projection to W_1/Σ is not fixed by Ξ, for some $a_2, \ldots, a_m \in \Sigma$ and for some E'-linearly independent $b_1, \ldots, b_m \in W_2$. Then there exists an element $h \in \Xi$ such that $ha_1 - a_1 \in W_1 \smallsetminus \Sigma$, and therefore, V contains an element of type $\sum_{i=1}^{m} a \otimes b_1$ for some $a \in W_1 \smallsetminus \Sigma$. $\qquad\square$

Proposition 2.6. *Let $W \in \mathcal{S}m_G(E)$ be an object. For any open subgroup U of G, denote by $W_{(U)}$ the sum of all proper subrepresentations of U in W; and by Ξ_U the pointwise stabilizer of $W_{(U)}$ in U. Suppose that for any open subgroup U of G:* (i) *the E-representation $W/W_{(U)}$ of U is absolutely irreducible and non-trivial in restriction to Ξ_U*[7] *and* (ii) *any irreducible smooth representation of K can be embedded into W so that its image does not meet $W_{(U)}$. Then $\mathcal{I}_S$ annihilates any quotient of $W \otimes_E V$ for any $V \in \mathcal{S}m_G(E)$.*

Proof. It suffices to check the vanishing of $\mathcal{I}_S(W \otimes_E V)$. Extending the coefficients if needed, we may assume that E is big enough (i.e., algebraically closed and $\#E > \#k$), so that any smooth irreducible E-representation of any open subgroup of G is absolutely irreducible.[8]

The vanishing holds if the G-module $W \otimes_E V$ is spanned by the elements $g\xi - \xi$ for all $\xi \in (W \otimes_E V)^K$ and all $g \in G$. Equivalently, as the restriction of V to K is semisimple, the G-span of such elements $g\xi - \xi$ contains $W \otimes_E \rho$ for any irreducible E-subrepresentation ρ of K in V. By (ii), W contains a E-subrepresentation of K which is (a) dual to ρ and (b) outside of $W_{(U)}$, where $U \subset G$ is the pointwise stabilizer of ρ. Then there is an element $\xi \in (W \otimes_E \rho)^K$, which is not in $W_{(U)} \otimes_E \rho$.

As the Ξ_U-module $W/W_{(U)}$ is non-trivial, there exists an element $u \in \Xi_U$ such that $\eta := u\xi - \xi$ is not in $W_{(U)} \otimes_E \rho$. Denote by $\widetilde{U}$ the subgroup in G generated by U and K. Then $\widetilde{U}$ contains U as a normal subgroup of finite index; $\widetilde{U}$ acts on $W_{(U)}$; ρ can be viewed as a representation of $\widetilde{U}$ via the identification $\widetilde{U}/U = K/U \cap K$. By Lemma 2.5 (with $H = \widetilde{U}$), the element η generates the $E[\widetilde{U}]$-module $W \otimes_E \rho$. $\qquad\square$

Lemma 2.7 (A source of representations of G containing all irreducible smooth representations of K). *If a subrepresentation W of G in $\bigotimes_F^\bullet \Omega^1_{F|k}$ does not contain regular forms,*[9] *i.e., forms from $\Gamma(X, \Omega^\bullet_{X|k})$ for a smooth proper k-variety X with $k(X) \subset F$, then W contains each irreducible smooth representation ρ of K.*

As mentioned in §2.1, if no non-zero element of $\mathbb{D}_E(G)$ annihilates a smooth representation W then W contains all irreducible smooth representations of K. The vanishing of the annihilators of F/k and $F^\times/k^\times$ is shown in [7, Prop.4.2]. Assume for simplicity that $F^K|k$ is purely transcendental.

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[6]Σ is H-invariant: as $\ker \rho$ is a normal subgroup of H, the group H permutes the $\ker \rho$-submodules in W_1, while Σ is the maximal proper $\ker \rho$-submodule in W_1.

[7]In particular, W is absolutely indecomposable. Any non-zero quotient of $A(F)$ for an absolutely simple algebraic k-group A is an example of such W. (Indeed, any open subgroup $U \subset G$ contains $G_{F|L}$ for a finitely generated L in $F|k$, so any $t \in A(F) \smallsetminus A(\overline{L})$ is a cyclic vector of $A(F)$, considered as U-module. Here $\overline{L}$ is the algebraic closure of L in F. If the transcendence degree of $L|k$ is minimal then, by [10], $\overline{L}$ is U-invariant, so $A(F)_{(U)} = A(\overline{L})$. $\qquad\square$)

[8]Schur's lemma=[2, Claim 2.11]: *Let H be a totally disconnected group and E be a field of cardinality greater than the cardinality of H/U for any open subgroup U of H. Then the endomorphisms of the smooth irreducible $\overline{E}$-representation of H are scalar.*

[9]Examples of such W are subrepresentations of $\mathrm{Sym}^2_F \Omega^1_{F|k}$, of $\Omega^\bullet_{F|k,\mathrm{exact}}$, or of the image in $\Omega^j_{F|k}$ of $\wedge^j \Omega^1_{F|k,\log}$, where $d\log : F^\times/k^\times \xrightarrow{\sim} \Omega^1_{F|k,\log}$, for any $j \geq 1$. It follows directly from Hilbert's Satz 90 that the representation F (and therefore, the irreducible representation $d : F/k \xrightarrow{\sim} \Omega^1_{F|k,\mathrm{exact}}$) of G contains all irreducible smooth (and thus, finite-dimensional) representations of K.

Proof. Let p_ρ be the central projector in the group algebra of $Q = K/\ker\rho$ onto the ρ-isotypical part. As explained in [8, Prop.7.6], W contains a non-zero element ω fixed by the group $G_{F|k(\mathbb{P}^M)}$ for an appropriate $M \geq 1$ and an embedding $k(\mathbb{P}^M) \hookrightarrow F$. The finite field extension $F^{\ker\rho}|F^K$ can be considered as a purely transcendental extension of a function field extension $k(Y)|k(Y)^Q$ of smooth projective k-varieties of dimension $\geq M$. Consider ω as a differential form with poles on $\mathbb{P}^M_k$. Fix a sufficiently general finite morphism $f : Y \longrightarrow \mathbb{P}^M_k$, unramified above the poles of ω, and such that the poles of $f^*\omega$ pass through a fixed point of Y, but not through another point of its Q-orbit. Then, as Q acts freely on the set of 'sufficiently general' divisors on Y, the form $p_\rho f^*\omega$ is non-zero, and thus, $p_\rho f^*\omega$ spans a K-submodule in W isomorphic to ρ. $\square$

Remark. The vanishing of $\mathcal{I}_S$ on any smooth semilinear representation V of G is evident: Let L be the function field of an affine k-space embedded into F. For any $v \in V^{G_{F|L}}$ and any $x \in F$ transcendental over L the vector xv belongs to $V^{G_{F|L(x)}}$, so its image $\mathcal{I}_S V$ should be fixed by G. In particular, the image of xv in $\mathcal{I}_S V$ coincides with the image of $2xv$, and thus, xv becomes 0 in $\mathcal{I}_S V$. Such vectors xv span V, so $\mathcal{I}_S V = 0$.

Corollary 2.8. *For any k-variety U and any rational closed form η on $U \times \mathbb{A}^1$ there exist an affine variety Y, dominant morphisms $\pi : Y \to U \times \mathbb{A}^1$, $\pi_1, \ldots, \pi_m : Y \to \mathbb{A}^N_k$ and rational closed forms $\eta_1, \ldots, \eta_m$ on $\mathbb{A}^N_k$ and η_0 on U such that $\pi^*\eta = (\mathrm{pr}_U \circ \pi)^*\eta_0 + \pi_1^*\eta_1 + \cdots + \pi_m^*\eta_m$.*

Proof. We consider η as a section of the sheaf $\underline{\Omega}^\bullet_{|k,\mathrm{closed}} : X \mapsto \Omega^\bullet_{k(X)|k,\mathrm{closed}}$ over $U \times \mathbb{A}^1$. Proposition 3.3 describes the kernel of $\underline{\Omega}^\bullet_{|k,\mathrm{closed}} \overset{\alpha}{\longrightarrow} \mathcal{I}\,\underline{\Omega}^\bullet_{|k,\mathrm{closed}}$ as the ideal generated by the exact and the logarithmic differentials. By Proposition 2.6, applied to $W = F^\times/k^\times$, $\mathcal{I}_S$ annihilates the kernel of α. Thus, modulo closed forms coming from projective spaces, η comes from U. $\square$

Let X be a smooth proper k-variety and $W := \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ be the module of generic 0-cycles on X. The space W^K is the image of the projector defined by the Haar measure of K. As the generators of W are generic points of X, the space W^K is spanned by the 0-cycles of type $p_*\pi^*q$ for all diagrams of dominant k-morphisms $X \overset{p}{\twoheadleftarrow} Y \overset{\pi}{\twoheadrightarrow} \mathbb{P}^N_k$, where π is generically finite, and all generic points $q \in \mathbb{P}^N(F^K)$. (Indeed, for any generic F-point $\sigma : k(X) \overset{/k}{\hookrightarrow} F$ of X the orbit $K\sigma$ is finite, so the compositum L_1 of the images of the elements of $K\sigma$ is finitely generated over k. Let $L_0 \subset F^K$ be a finitely generated and purely transcendental extension of k containing L_1^K. Let Y be a K-equivariant smooth k-model of $L_0 L_1$. Then p and π are induced by the inclusions $k(X) \subset k(Y) \supset L_0$.) Thus, the module $\mathcal{I}_S W$ is the quotient of W by the $\mathbb{Q}$-span of 0-cycles of type $p_*\pi^*q_1 - p_*\pi^*q_2$ for all dominant k-morphisms $p : Y \to X$, generically finite k-morphisms $\pi : Y \to \mathbb{P}^N_k$ and all generic points $q_1, q_2 \in \mathbb{P}^N(F)$.

Lemma 2.9. *Let X be a smooth proper curve over k of genus g. Then the G-module $Z_0^{\mathrm{rat}}(k(X) \otimes_k F) := \ker[Z_0(k(X) \otimes_k F) \longrightarrow CH_0(X \times_k F)]$ is generated by $w_N = \sum_{j=1}^N \sigma_j - \sum_{j=1}^N \tau_j$ for all $N > g$, where $(\sigma_1, \ldots, \sigma_N; \tau_1, \ldots, \tau_N)$ is a generic F-point of the fibre over 0 of the map $X^N \times_k X^N \overset{p_N}{\longrightarrow} \mathrm{Pic}^0 X$ sending $(x_1, \ldots, x_N; y_1, \ldots, y_N)$ to the class of $\sum_{j=1}^N x_j - \sum_{j=1}^N y_j$.*

Proof. Let $\gamma_1, \ldots, \gamma_s : k(X) \overset{/k}{\hookrightarrow} F$ and $\delta_1, \ldots, \delta_s : k(X) \overset{/k}{\hookrightarrow} F$ be generic points of X such that $\sum_{j=1}^s \gamma_j - \sum_{j=1}^s \delta_j$ is the divisor of a rational function on X_F.

We need to show that $\sum_{j=1}^s \gamma_j - \sum_{j=1}^s \delta_j$ belongs to the G-submodule in $Z_0^{\mathrm{rat}}(k(X) \otimes_k F)$ generated by w_N's.

There is a collection $\alpha_1, \ldots, \alpha_g : k(X) \overset{/k}{\hookrightarrow} F$ of generic points of X such that the class of $\sum_{j=1}^s \gamma_j + \sum_{j=1}^g \alpha_j$ in $\mathrm{Pic}^{s+g} X$ is a generic point. Then there is a collection $\xi_1, \ldots, \xi_{s+g} : k(X) \overset{/k}{\hookrightarrow} F$ of generic points of X in general position such that $\sum_{j=1}^s \gamma_j + \sum_{j=1}^g \alpha_j - \sum_{j=1}^{s+g} \xi_j$ is divisor of a rational function on X_F (so the same holds also for $\sum_{j=1}^s \delta_j + \sum_{j=1}^g \alpha_j - \sum_{j=1}^{s+g} \xi_j$). We may, thus, assume that $\delta_1, \ldots, \delta_s$ are in general position.

Fix a collection $\{\varkappa_{ij}\}_{1 \leq i \leq g, 1 \leq j \leq s}$ of generic points of X in general position, also with respect to $\gamma_1, \ldots, \gamma_s$ and to $\delta_1, \ldots, \delta_s$, such that the classes of $\gamma_1 + \sum_{i=1}^g \varkappa_{i1}, \ldots, \gamma_s + \sum_{i=1}^g \varkappa_{is}$ in $\mathrm{Pic}^{g+1} X$ are generic points in general position. Then one can choose a collection $\{\xi_{ij}\}_{0 \leq i \leq g, 1 \leq j \leq s}$ of generic points of X in general

position such that $\gamma_j + \sum_{i=1}^{g} \varkappa_{ij} - \sum_{i=0}^{g} \xi_{ij}$ is divisor of a rational function on X_F (so the same holds also for $\sum_{j=1}^{s} \sum_{i=0}^{g} \xi_{ij} - \left(\sum_{j=1}^{s} \delta_j + \sum_{j=1}^{s} \sum_{i=1}^{g} \varkappa_{ij} \right)$). We may, thus, assume that both $\gamma_1, \ldots, \gamma_s$ and $\delta_1, \ldots, \delta_s$ are in general position.

Then there is a collection of generic points $\xi_1, \ldots, \xi_s : k(X) \overset{/k}{\hookrightarrow} F$ such that the points $(\gamma_1, \ldots, \gamma_s; \xi_1, \ldots, \xi_s)$ and $(\delta_1, \ldots, \delta_s; \xi_1, \ldots, \xi_s)$ are generic on $p_s^{-1}(0)$. Then $\sum_{j=1}^{s} \gamma_j - \sum_{j=1}^{s} \xi_j$ and $\sum_{j=1}^{s} \delta_j - \sum_{j=1}^{s} \xi_j$ are divisors of rational functions on X_F. Clearly, such elements belong to the G-orbit of w_s. $\qquad\square$

Remark. The G-module $Z_0^{\mathrm{rat}}(k(X) \otimes_k F)$ from Lemma 2.9 is generated by w_{g+1}. *Proof.* There exists an effective divisor D (of degree g) in the linear equivalence class of $\sum_{j=2}^{N} \sigma_j - \sum_{j=g+2}^{N} \tau_j$, so $w_N = [\sum_{j=2}^{N} \sigma_j - D - \sum_{j=g+2}^{N} \tau_j] + [\sigma_1 + D - \sum_{j=1}^{g+1} \tau_j]$ is a sum of an element in the G-orbit of w_{N-1} and an element in the G-orbit of w_{g+1}. $\qquad\square$

Proposition 2.10. $\mathcal{I}_S \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}] = \mathrm{Pic}(X_F)_{\mathbb{Q}}$ *for any smooth proper curve X over k.*

Proof. By Lemma 2.9, it suffices to show that the images of the generators w_N in $\mathcal{I}_S W$ are zero. Denote by $g \geq 0$ the genus of X, by ψ_N a generic effective divisor on X of degree N with a special class in $\mathrm{Pic}^N X$. Then $w_N = \sigma \psi_{N+g} - \tau \psi_{N+g}$ for some $\sigma, \tau \in G$, so it suffices to show that the images of ψ_N's in $\mathcal{I}_S W$ are fixed by G. Denote by $X^N \overset{s}{\longrightarrow} \Sigma^N X \overset{r}{\longrightarrow} \mathrm{Pic}^N(X)$ the natural morphisms and set $Y := (rs)^{-1}(\ast)$. Let $p : Y \subseteq X^N \longrightarrow X$ be the projection to the first multiple; set $\pi = s|_Y : Y \longrightarrow r^{-1}(\ast)$. The projection to the first $N - g$ multiples $Y \longrightarrow X^{N-g}$ is generically finite of degree $g!$. If $N \geq 2g - 1$ then $r^{-1}(\ast) \cong \mathbb{P}^{N-g}$. Assume also that $N \geq g + 1$ (i.e. $N \geq \max(2g - 1, g + 1)$). As s is generically finite of degree $N!$, one has $(N - 1)! \psi_N = p_* \pi^* q$ for a generic point q of $r^{-1}(\ast)$. $\qquad\square$

Denote by $\mathcal{I}_G$ the full subcategory in $\mathcal{S}m_G$ of "homotopy invariant" representations: $W^{G_{F|L'}} = W^{G_{F|L}}$ for any purely transcendental subextension $L'|L$ in $F|k$.

Theorem 2.11. *A dominant sheaf is $\mathbb{A}^1$-invariant if and only if all its simple subquotients are.*

Proof. Let S be the collection of pairs of type $(G_{F|L(x)} \subset G_{F|L})$ for all subfields L in $F|k$ of finite type and elements $x \in F$ transcendental over L. The following conditions are equivalent:
   (1) a smooth representation W of G is "homotopy invariant";
   (2) $W^{U_1} = W^{U_2}$ for all pairs $(U_1 \subset U_2) \in S$;
   (3) $W^{G_{F|L}} = W^{G_{F|L'}}$ for all subfields L in $F|k$ of finite type and purely transcendental extensions $L'|L$ in F such that F is algebraic over L'.

(1)$\Leftrightarrow$(3) and (1)$\Leftrightarrow$(2) are evident; (2)$\Leftrightarrow$(1) is proved in [7, Corollary 6.2]. This verifies the condition (2) of Proposition 2.2. For each pair $(G_{F|L(x)} \subset G_{F|L}) \in S$ fix some $\sigma \in G_{F|L}$ with x and σx algebraically independent over L. Then the condition (1)(i) is obvious: $G_{F|L(x)} \cap G_{F|L(\sigma x)} = G_{F|L(x, \sigma x)}$ and $(G_{F|L(x, \sigma x)} \subset G_{F|L(x)}) \in S$; the condition (1)(ii) follows from [7, Lemma 2.16]: the subgroups $G_{F|L(x)}$ and $G_{F|L(\sigma x)}$ generate $G_{F|L}$. $\qquad\square$

2.5. **Summary of equivalences.** The following categories are equivalent:
   (1) the category of dominant $\mathbb{A}^1$-invariant sheaves of E-vector spaces;
   (2) the category of dominant $\mathbb{A}^1$-invariant presheaves of E-vector spaces with the Galois descent property;
   (3) the category $\mathcal{S}m_G^S(E)$, where S consists of the pairs of type $(G_{F|L'} \subset G_{F|L})$ with purely transcendental $L'|L$ in $F|k$.

These equivalences restrict to equivalences of corresponding subcategories: (1) of sheaves of finite-dimensional spaces, (2) of presheaves of finite-dimensional spaces, (3) of admissible representations of G.[10]

Consider the following properties of a smooth representation W of G:
   (1) $W \in \mathcal{S}m_G^S(E)$, where S consists of the pairs of type $(G_{F|L'} \subset G_{F|L})$ with purely transcendental $L'|L$ in $F|k$;

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[10]A representation of a totally disconnected group is *admissible* if it is smooth and the fixed subspaces of all open subgroups are finite-dimensional.

(2) the restriction of W to a compact subgroup U does not contain all smooth irreducible representations of U;

(3) the annihilator of W in the algebra $\mathbb{D}_{\mathbb{Q}}(G)$ is non-zero.

One has (1)$\Rightarrow$(2)$\Rightarrow$(3). [(2)$\Rightarrow$(3) is explained in §2.1. (1)$\Rightarrow$(2): If F^U is purely transcendental over k, there are many irreducible smooth representations of U, entering in no object of $\mathcal{S}m_G^S(E)$. Any non-trivial smooth irreducible representation τ of U such that $F^{\ker \tau}$ is unirational (e.g., purely transcendental) over k is an example of such representation. Clearly, for any such τ the natural projector $p_\tau \in \mathbb{D}_{\mathbb{Q}}(G)$ onto the τ-isotypical part belongs to the common annihilator of the objects of $\mathcal{S}m_G^S(E)$.]

Remark. For a discrete valuation v of rank 1 on F, trivial on k,[11] and a smooth representation W of G set $W_v := \sum_L W^{G_{F|L}} \subseteq W$, where L runs over the subfields in the valuation ring of v. The intersection $\Gamma(W) := \bigcap_v W_v$ over all such v's is again in $\mathcal{S}m_G$. As shown in [10, Cor.4.7], the property (1) for W implies that $W = W_v$ (and also $W = \Gamma(W)$, since all v's as above form a G-orbit, cf. [10]).

3. Differential forms

Let $H^\bullet = \bigoplus_{q \geq 0} H^q$ be a cohomology theory, considered as a dominant $\mathbb{A}^1$-presheaf. Denote by $\underline{H}_c^\bullet$ the dominant $\mathbb{A}^1$-sheaf $X \mapsto H^\bullet(X)/N^1$ for smooth proper k-varieties X, which is a subsheaf of $\underline{H}^\bullet$, e.g., $\underline{H}_c^1 : X \mapsto H^1(\overline{X})$. Clearly, $\underline{H}_c^\bullet$ is a sheaf of finite $H^\bullet(k)$-modules. It would follow from the standard semisimplicity conjecture that the sheaf $\underline{H}_c^\bullet$ is semisimple if $H^\bullet(k)$ is a field.

We shall be interested in the case of de Rham cohomology $H^\bullet = H_{\mathrm{dR}/k}^\bullet : X \mapsto H_{\mathrm{dR}/k}^\bullet(X) := \mathbb{H}^\bullet(X, \Omega_{X|k}^\bullet)$, where $H^\bullet(k) = k$, cf. [3]. Clearly, $\underline{H}_{\mathrm{dR}/k}^q = \Omega_{|k,\mathrm{closed}}^q / \Omega_{|k,\mathrm{exact}}^q$, where $\Omega_{|k,\mathrm{closed}}^q : Y \mapsto \ker(d|\Gamma(Y, \Omega_{Y|k}^q))$ and $\Omega_{|k,\mathrm{exact}}^q : Y \mapsto d\Gamma(Y, \Omega_{Y|k}^{q-1})$, so $d : \mathcal{H}_1^{\mathbb{G}_a} \xrightarrow{\sim} \Omega_{|k,\mathrm{exact}}^1$. The sheaf $\underline{H}_{\mathrm{dR}/k,c}^1$ is semisimple. It is described in Lemma 1.1.

3.1. Maximal $\mathbb{A}^1$-subsheaf and the $\mathbb{A}^1$-quotient of (closed) forms.
Recall (§1.2) that the inclusion functor $\mathcal{I}_G \to \mathcal{S}m_G$ admits a right adjoint $W \mapsto W^{(0)}$, the maximal subobject in $\mathcal{I}_G$. The following fact points out once more the cohomological nature of the objects of $\mathcal{I}_G$.

Proposition 3.1 ([8], Prop.7.6). *The maximal subobject in $\mathcal{I}_G$ of the sheafification of $\bigotimes_{\mathcal{O}}^\bullet \Omega_{|k}^1$ is $\Omega_{|k,\mathrm{reg}}^\bullet$.*

For any smooth proper k-variety Y there are the following canonical isomorphisms

$$(3) \qquad \mathrm{Hom}_{\mathcal{I}_G}(C_{k(Y)}, \Omega_{|k,\mathrm{reg}}^q) = \mathrm{Hom}_{\mathcal{S}m_G}(\Psi_Y, \Omega_{|k,\mathrm{reg}}^q) \xleftarrow{\sim} \Gamma(Y, \Omega_{Y|k}^q) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}m_G}(CH_0(Y_F), \Omega_{|k,\mathrm{reg}}^q).$$

The first isomorphism is functorial with respect to the dominant morphisms $Y \longrightarrow Y'$, the second one is functorial with respect to arbitrary morphisms $Y \longrightarrow Y'$.

Lemma 3.2. *Let L be an algebraically closed extension of k and x be an indeterminant. Then there are isomorphisms $\mathrm{id} + \sum_{\alpha \in L} \wedge \frac{d(x-\alpha)}{x-\alpha} : (L(x) \otimes_L \Omega_{L|k}^q)/\Omega_{L|k,\mathrm{exact}}^q \oplus \bigoplus_{\alpha \in L} (\Omega_{L|k}^{q-1}/\Omega_{L|k,\mathrm{exact}}^{q-1}) \xrightarrow{\sim} \Omega_{L(x)|k}^q/\Omega_{L(x)|k,\mathrm{exact}}^q$ and $d + \sum_{\alpha \in L} \wedge \frac{d(x-\alpha)}{x-\alpha} : (L(x) \otimes_L \Omega_{L|k}^q)/\Omega_{L|k,\mathrm{closed}}^q \oplus \bigoplus_{\alpha \in L} \Omega_{L|k,\mathrm{exact}}^q \xrightarrow{\sim} \Omega_{L(x)|k,\mathrm{exact}}^{q+1}$ for any $q \geq 1$. The former isomorphism restricts to an isomorphism $H_{\mathrm{dR}/k}^q(L) \oplus \bigoplus_{\alpha \in L} H_{\mathrm{dR}/k}^{q-1}(L) \xrightarrow{\sim} H_{\mathrm{dR}/k}^q(L(x))$.*

Proof. As $\Omega_{L(x)|k}^q = L(x) \otimes_L \Omega_{L|k}^q \oplus L(x) \otimes_L \Omega_{L|k}^{q-1} \wedge dx$, for any $\omega \in \Omega_{L(x)|k}^q$ one has $\omega \equiv \eta \wedge dx$ (mod $L(x) \otimes_L \Omega_{L|k}^q$) for a unique $\eta \in L(x) \otimes_L \Omega_{L|k}^{q-1}$. Using partial fraction decomposition of rational functions in $L(x)$, we get a presentation $\eta = \sum_{j \geq 0} x^j \eta_j + \sum_{\alpha \in L, \, j \geq 1} \frac{\eta_{j,\alpha}}{(x-\alpha)^j}$, where $\eta_j, \eta_{j,\alpha} \in \Omega_{L|k}^{q-1}$. Then $\eta \wedge dx \equiv \sum_{\alpha \in L} \eta_{1,\alpha} \wedge \frac{d(x-\alpha)}{x-\alpha}$ (mod $L(x) \otimes_L \Omega_{L|k}^q + \Omega_{L(x)|k,\mathrm{exact}}^q$), so $\omega \equiv \sum_i \phi_i(x) \eta_i + \sum_{\alpha \in L} \eta_{1,\alpha} \wedge \frac{d(x-\alpha)}{x-\alpha}$ (mod $\Omega_{L(x)|k,\mathrm{exact}}^q$), and thus, $d\omega = \sum_i d\phi_i(x) \wedge \eta_i + \sum_i \phi_i(x) d\eta_i + \sum_{\alpha \in L} d\eta_{1,\alpha} \wedge \frac{d(x-\alpha)}{x-\alpha} \equiv \sum_i \phi_i'(x) dx \wedge \eta_i + \sum_{\alpha \in L} d\eta_{1,\alpha} \wedge \frac{dx}{x-\alpha}$ (mod $L(x) \otimes_L \Omega_{L|k}^q$) for some $\phi_i(x) \in L(x)$ and $\eta_i \in \Omega_{L|k}^q$ (and we may assume that η_i are L-linearly independent). Using partial fraction decomposition of the rational functions $\phi_i \in L(x)$, we see that if ω is closed then $d\eta_{1,\alpha} = 0$, $\phi_i \in L$ and $\sum_i \phi_i \eta_i \in \Omega_{L|k}^q$ is closed. $\square$

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[11]By definition, this means that any maximal system of elements of $F^\times$ with independent images in the valuation group, should be a transcendence base of F over a lift of a subfield of the residue field.

Proposition 3.3. *Let M_q be the sheaf associated with the presheaf $\Omega^q_{|k,\mathrm{exact}} + \Omega^{q-1}_{|k,\mathrm{closed}} \wedge d\log \mathbb{G}_m \subset \Omega^q_{|k}$ for any $q \geq 1$. Then* (i) $\Omega^q_{k(X \times \mathbb{A}^n)|k,\mathrm{closed}} = \Omega^q_{k(X)|k,\mathrm{closed}} + M_q(X \times \mathbb{A}^n)$ *for any $n \geq 1$;* (ii) M_q *is the kernel of the natural projection $\pi_q : \underline{\Omega}^q_{|k,\mathrm{closed}} \to V^q := \mathcal{I}(\underline{\Omega}^q_{|k,\mathrm{closed}}) = \mathcal{I}(\underline{H}^q_{\mathrm{dR}|k})$;* (iii) *for $q \geq 2$, M_q is the sheaf associated with the presheaf $\Omega^{q-1}_{|k,\mathrm{closed}} \wedge d\log \mathbb{G}_m$ and $d + d\log : \mathcal{H}^{\mathbb{G}_a}_1 \oplus k \otimes \mathcal{H}^{\mathbb{G}_m}_1 \to M_1$ is an isomorphism.*

In particular, the natural projections $\underline{\Omega}^\bullet_{|k,\mathrm{closed}} \overset{p_1}{\to} \underline{H}^\bullet_{\mathrm{dR}|k} \overset{p_2}{\to} V^\bullet := \mathcal{I}(\underline{H}^\bullet_{\mathrm{dR}|k}) = \mathcal{I}(\underline{\Omega}^\bullet_{|k,\mathrm{closed}})$ are morphisms of sheaves of supercommutative k-algebras. (The kernel of p_1, i.e. $\underline{\Omega}^\bullet_{|k,\mathrm{exact}}$, is the ideal generated by $\underline{\Omega}^1_{|k,\mathrm{exact}}$,[12] the kernel of p_2 is the ideal generated by $d\log \mathbb{G}_m$.) They are surjective even as morphisms of presheaves.

Proof. Let us show that $\ker \pi_q$ contains M_q. For any irreducible smooth k-variety X, any $\eta \in \Omega^{q-1}_{k(X)|k,\mathrm{closed}}$ and a generator t of the field $k(X \times \mathbb{G}_m)$ over $k(X)$ the closed q-forms $\omega_m = \eta \wedge d\log t$ and $\omega_a = \eta \wedge dt$ are sections of the sheaf $\Omega^q_{|k,\mathrm{closed}}$ over $X \times \mathbb{G}_m$, so their images in $\mathcal{I}(\Omega^q_{|k,\mathrm{closed}})$ should be sections over X. As there are endomorphisms g_m, g_a of $X \times \mathbb{G}_m | X$ such that $g_m t = t^2$ and $g_a t = 2t$ (so $g_? \omega_? = 2\omega_?$), the images of $\omega_?$ in $\mathcal{I}(\Omega^q_{|k,\mathrm{closed}})$ should be zero. The elements of type $\eta \otimes d\log t$ (resp., $\eta \otimes dt$) span the sheaf $\Omega^{q-1}_{|k,\mathrm{closed}} \otimes \mathcal{H}^{\mathbb{G}_m}_1$ (resp., $\Omega^{q-1}_{|k,\mathrm{closed}} \otimes \mathcal{H}^{\mathbb{G}_a}_1$, which is surjective over $\Omega^q_{|k,\mathrm{exact}}$).

By [7, Lemma 6.3, p.200], to show that $\ker \pi_q = M_q$ it suffices to check that, for any algebraically closed extension $F'|k$ in F and any $t \in F \setminus F'$, any $\omega \in \Omega^q_{F'(t)|k,\mathrm{closed}}$ belongs in fact to $\Omega^q_{F'|k,\mathrm{closed}} + M_q$.

By Lemma 3.2, $\omega \equiv \xi + \sum_{\alpha \in F'} \eta_\alpha \wedge \frac{d(t-\alpha)}{t-\alpha}$ $(\mathrm{mod}\ \Omega^q_{F'(t)|k,\mathrm{exact}})$, where $\xi \in \Omega^q_{F'|k,\mathrm{closed}}$ and $\eta_\alpha \in \Omega^{q-1}_{F'|k,\mathrm{closed}}$, which means that $\omega \in \Omega^q_{F'|k,\mathrm{closed}} + M_q$. $\square$

Conjecture 3.4. *The sheaf $V^\bullet$ is semisimple.*

Remarks. 1. It follows from Proposition 3.3 that the natural morphism $\underline{H}^\bullet_{\mathrm{dR}/k,c} \to V^\bullet$ is injective.

2. As explained in Remark on p.10, $\mathcal{I}V = 0$ for any semilinear smooth representation V: if $v \in V^{G_{F|L}}$ and $f \in F$ is transcendental over L then $v = fv - (f-1)v$ becomes zero in any quotient of V in $\mathcal{I}_G$.

3. For an algebra $A \in \mathcal{S}\mathrm{m}_G$ it is not always true that the kernel A° of the projection $A \to \mathcal{I}A$ is an ideal. E.g., let $A = A_\bullet$ be the (graded) tensor, symmetric or skew-symmetric algebra of $A_1 = \mathbb{Q}[F \setminus k]$. Then $\mathcal{I}A_1 = \mathbb{Q}$, so $A_1^\circ \otimes A_1 + A_1 \otimes A_1^\circ$ consists of all sums in $\mathbb{Q}[(F \setminus k) \times (F \setminus k)]$ of degree 0. On the other hand, $\mathcal{I}(A_1 \otimes A_1) = \bigoplus_{x \in \mathbf{Spec}(k(\mathbb{P}^1) \otimes_k k(\mathbb{P}^1))} C_{k(x)}$, and therefore, $A_1^\circ \otimes A_1 + A_1 \otimes A_1^\circ$ is strictly bigger than A_2°.

3.2. **The semisimplicity of the regular forms of top degree.** Let L be an algebraically closed extension of k with $1 \leq q = \mathrm{tr.deg}(L|k) < \infty$. Define a representation $\Omega^q_{L|k,\mathrm{reg}}$ as the union in $\Omega^q_{L|k}$ of all spaces $\Gamma(X, \Omega^q_{X|k})$ over all smooth proper varieties X over k with the function field embedded into L over k.

The naïve truncation filtration on $\Omega^\bullet_{\overline{U}|k}$ gives the descending Hodge filtration $F^\bullet$ on $H^q_{\mathrm{dR}/k}(\overline{U})$. The Hodge filtrations on $H^q_{\mathrm{dR}/k}(\overline{U})$ for all U's induce a canonical filtration $F^\bullet$ on $\underline{H}^q_{\mathrm{dR}/k,c}$ by subsheaves of k-vector spaces with associated graded quotients $H^{p,q-p}_{|k} : Y \mapsto \mathrm{coker}[\bigoplus_D H^{p-1}(D, \Omega^{q-p-1}_{D|k}) \longrightarrow H^p(\overline{Y}, \Omega^{q-p}_{\overline{Y}|k})]$, where $D \to \overline{Y}$ runs over all resolutions of the divisors on $\overline{Y}$. In particular, $H^{q,0}_{|k} = F^q \underline{H}^q_{\mathrm{dR}/k,c} = \Omega^q_{|k,\mathrm{reg}} : Y \mapsto \Gamma(\overline{Y}, \Omega^q_{\overline{Y}|k})$ is the dominant subsheaf of $\underline{H}^q_{\mathrm{dR}/k,c}$ consisting of regular differential q-forms.

Proposition 3.5 ([4])**.** *Suppose that the cardinality of k is at most continuum. The representation $H^q_{\mathrm{dR}/k,c}(L)$ (and therefore, $\Omega^q_{L|k,\mathrm{reg}}$) of $G_{L|k}$ is semisimple. Any embedding $\iota : k \hookrightarrow \mathbb{C}$ into the field of complex numbers determines*

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[12]More generally, let $\omega \in \Omega^{\geq i}_{F|k} \setminus \Omega^{\geq i+1}_{F|k}$ be a closed form for some $i \geq 0$. Then any ideal in $\underline{\Omega}^\bullet_{|k,\mathrm{closed}}$ containing the G-orbit of ω contains $\underline{\Omega}^{\geq i+1}_{|k,\mathrm{exact}}$. *Proof.* By [8, Lemma 7.7], the semilinear representation $\Omega^j_{F|k}$ is irreducible for any $j \geq 0$. In particular, F-linear envelope of the G-orbit of ω is the direct sum of $\Omega^j_{F|k}$ over all $j \geq i$ such that the homogeneous component of ω of degree j is non-zero. Then $dz \wedge \sigma\omega = d(z \cdot \sigma\omega)$ for all $z \in F$ and all $\sigma \in G$ span the direct sum of $\underline{\Omega}^j_{|k,\mathrm{exact}}$ over all $j \geq i$ as above. $\square$

- a $\mathbb{C}$-antilinear isomorphism $H_{L|k}^{s,t} \otimes_{k,\iota} \mathbb{C} \cong H_{L|k}^{t,s} \otimes_{k,\iota} \mathbb{C}$,
- a positive definite $G_{L|k}$-equivariant hermitian form $(\mathbb{C} \otimes_{k,\iota} H_{\mathrm{dR}/k,c}^q(L)) \otimes_{id,\mathbb{C},\sigma} (\mathbb{C} \otimes_{k,\iota} H_{\mathrm{dR}/k,c}^q(L)) \longrightarrow$ $\mathbb{C}(\chi)$, where σ is the complex conjugation and χ is the modulus of $G_{L|k}$.

There exists a non-canonical $\mathbb{Q}$-linear isomorphism $H_{L|k}^{s,t} \cong H_{L|k}^{t,s}$.

Proof. For any smooth projective k-variety X the complexified projection $F^p H_{\mathrm{dR}/k}^{p+q}(X) \to H^q(X, \Omega_{X|k}^p)$ identifies $F^p H_{\mathrm{dR}/k}^{p+q}(X) \otimes_{k,\iota} \mathbb{C} \cap \overline{F^q H_{\mathrm{dR}/k}^{p+q}(X) \otimes_{k,\iota} \mathbb{C}}$ with $H^q(X, \Omega_{X|k}^p) \otimes_{k,\iota} \mathbb{C}$. This gives a decomposition $\mathbb{C} \otimes_{k,\iota} H_{\mathrm{dR}/k,c}^q(L) = \bigoplus_{s+t=q} \mathbb{C} \otimes_{k,\iota} H_{L|k}^{s,t}$. Then the complex conjugation on $H^{p+q}(X_\iota(\mathbb{C}), \mathbb{C}) = H^{p+q}(X_\iota(\mathbb{C}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ identifies $H^q(X, \Omega_{X|k}^p) \otimes_{k,\iota} \mathbb{C}$ with $H^p(X_\iota(\mathbb{C}), \Omega_{X_\iota(\mathbb{C})}^q) = H^p(X, \Omega_{X|k}^q) \otimes_{k,\iota} \mathbb{C}$.

The semisimplicity of the k-representation $H_{\mathrm{dR}/k,c}^q(L)$ of $G_{L|k}$ is equivalent to the semisimplicity of its complexification. For the latter note that there is a positive definite $G_{L|k}$-equivariant hermitian form $(\mathbb{C} \otimes_{k,\iota} H_{L|k}^{s,t}) \otimes_{id,\mathbb{C},\sigma} (\mathbb{C} \otimes_{k,\iota} H_{L|k}^{s,t}) \longrightarrow \mathbb{C}(\chi)$, given by $(\omega, \eta) = \int_{X_\iota(\mathbb{C})} i^{q^2+2t} \omega \wedge \overline{\eta} \cdot [G_{L|k(X)}]$ for any $\omega, \eta \in H_{\mathrm{prim}}^{s,t}(X_\iota(\mathbb{C})) = H_{\mathrm{prim}}^t(X, \Omega_{X|k}^s) \otimes_{k,\iota} \mathbb{C} \subset \mathbb{C} \otimes_{k,\iota} H_{L|k}^{s,t}$. Here $H_{\mathrm{prim}}^{s,t}(X_\iota(\mathbb{C}))$ denotes the subspace orthogonal to the sum of all Gysin maps $H^{s-1,t-1}(D) \longrightarrow H^{s,t}(X_\iota(\mathbb{C}))$ for all desingularizations D of all divisors on $X_\iota(\mathbb{C})$, as in the definition of $\Omega_{L|k,\mathrm{reg}}^q$, X runs over all smooth proper k-varieties with the function field embedded into $L|k$. $\qquad \square$

3.3. Structure of closed 1-forms.
Let $\mathrm{Div}_{\mathbb{Q}}^\circ : Y \mapsto \mathrm{Div}_{\mathrm{alg}}(\overline{Y})_{\mathbb{Q}}$ be the presheaf of algebraically trivial divisors. It is a sheaf.

Lemma 3.6. *The residue homomorphism* $\mathrm{Res}_Y : H_{\mathrm{dR}/k}^1(k(Y)) \to k \otimes \mathrm{Div}(\overline{Y})$, $\omega \mapsto (\mathrm{res}_x\omega)_{x \in \overline{Y}^1}$, *defines a morphism of sheaves* $\mathrm{Res} : \underline{H}_{\mathrm{dR}/k}^1 \to k \otimes \mathrm{Div}_{\mathbb{Q}}^\circ$. *The short sequence* $0 \to \underline{H}_{\mathrm{dR}/k,c}^1 \to \underline{H}_{\mathrm{dR}/k}^1 \xrightarrow{\mathrm{Res}} \mathrm{Div}_{\mathbb{Q}}^\circ \otimes k \to 0$ *is exact, even as a sequence of presheaves.*

Proof. As Res commutes with the restriction to any sufficiently general curve C, $\mathrm{Res}_X(\omega) \cdot C = \mathrm{Res}_C(\omega|_C) \in CH_0(X)$, $\deg(\mathrm{Res}_X(\omega) \cdot C) = 0$ by Cauchy theorem, the pairing $\mathrm{NS}(X)_{\mathbb{Q}} \otimes CH_1(X)_{\mathbb{Q}}/hom \longrightarrow \mathbb{Q}$ is non-degenerate (by Lefschetz hyperplane section theorem), the class of $\mathrm{Res}_X(\omega)$ in $\mathrm{NS}(X)_{\mathbb{Q}}$ is zero. Thus, Res_X factors through the algebraically trivial divisors on X.

Clearly, the kernel of Res coincides with $\underline{H}_{\mathrm{dR}/k,c}^1$, cf. [6].[13] Then it remains to show that any algebraically trivial divisor on X is the residue of a closed 1-form. Any algebraically trivial divisor can be written as $D_1 - D_2$ for a pair D_1, D_2 of algebraically equivalent effective divisors on X. There is a smooth projective curve C, and an effective divisor D on $X \times C$, such that $\mathrm{pr}_X : D \to X$ is generically finite and $D_P - D_Q = D_1 - D_2$ for some points $P, Q \in C$. By Riemann–Roch theorem for curves, there exists a 1-form $\omega_{P,Q} \in \Omega_C^1(P + Q)$ such that $\mathrm{Res}_C(\omega_{P,Q}) = P - Q$: there is a non-holomorphic 1-form with simple poles in the set $\{P, Q\}$, since $\dim_k \Gamma(C, \Omega_C^1(P + Q)) = \dim_k \Gamma(C, \Omega_C^1) + 1$; there are no 1-forms with precisely one simple pole, since $\Gamma(C, \Omega_C^1(P)) = \Gamma(C, \Omega_C^1(Q)) = \Gamma(C, \Omega_C^1)$. Then $\mathrm{Res}_X(\mathrm{pr}_{X*}((\mathrm{pr}_C^*\omega_{P,Q})|_D)) = D_1 - D_2$. $\qquad \square$

Proposition 3.7.
- *The maximal semisimple subsheaf of* $\underline{\Omega}_{|k,\mathrm{closed}}^1$ *is canonically isomorphic to the direct sum* $\bigoplus_A \Gamma(A, \Omega_{A|k}^1)^{A(k)} \otimes_{\mathrm{End}(A)} \mathcal{H}_1^A = \mathcal{H}_1^{\mathbb{G}_a} \oplus k \otimes \mathcal{H}_1^{\mathbb{G}_m} \oplus \Omega_{|k,\mathrm{reg}}^1$, *where* A *runs over the set of isogeny classes of simple commutative algebraic k-groups;* $\Gamma(A, \Omega_{A|k}^1)^{A(k)} = \mathrm{Hom}_k(\mathrm{Lie}(A), k)$ *denotes the space of translation invariant 1-forms on* A. *The projection* $\underline{\Omega}_{|k,\mathrm{closed}}^1 \to \underline{\Omega}_{|k,\mathrm{closed}}^1 / \Omega_{|k,\mathrm{reg}}^1$ *is split (but not canonically).*
- *The maximal semisimple subsheaf of* $\underline{H}_{\mathrm{dR}/k}^1$ *is canonically isomorphic to* $\bigoplus_A H_{\mathrm{dR}/k}^1(A) \otimes_{\mathrm{End}(A)} \mathcal{H}_1^A = k \otimes \mathcal{H}_1^{\mathbb{G}_m} \oplus \underline{H}_{\mathrm{dR}/k,c}^1$, *where* A *runs over the set of isogeny classes of simple commutative algebraic k-groups (with the zero summand corresponding to $\mathbb{G}_a$). The projection* $\underline{H}_{\mathrm{dR}/k}^1 \to \underline{H}_{\mathrm{dR}/k}^1 / \underline{H}_{\mathrm{dR}/k,c}^1$ *is split (but not canonically).*

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[13]If the residues of $\omega \in H_{\mathrm{dR}/k}^1(k(X))$ are zero then integration along a loop depends only on its homology class in $H_1(X, \mathbb{Q})$. There is an element η of $H_{\mathrm{dR}/k}^1(X)$ with the same periods as ω, so integration of $\omega - \eta$ along a path joining a fixed (rational) point with the variable one is independent of a chosen path, and defines a meromorphic (i.e. rational) function.

- *The sheaf $V^1 : Y \mapsto H^1_{\mathrm{dR}/k}(k(Y))/k \otimes (k(Y)^\times/k^\times)$ from Proposition 3.3 is canonically isomorphic to $\bigoplus_A V^1(A) \otimes_{\mathrm{End}(A)} \mathcal{H}_1^A$, where A runs over the set of isogeny classes of simple abelian k-varieties.*

For any integer $q \geq 1$, the representation $\Omega^1_{L|k,\mathrm{closed}}$ of the group $G_{L|k}$ admits similar description (cf. §3.2).

Proof. In notation of Lemma 3.6, the sheaf $\mathrm{Div}^\circ_\mathbb{Q}$ admits a natural surjective morphism onto the Picard sheaf $\mathrm{Pic}^\circ_\mathbb{Q} = \mathrm{coker}[\mathcal{H}_1^{\mathbb{G}_m} \xrightarrow{\mathrm{div}} \mathrm{Div}^\circ_\mathbb{Q}] : Y \mapsto \mathrm{Pic}^0(\overline{Y})_\mathbb{Q}$ with the irreducible kernel $\mathcal{H}_1^{\mathbb{G}_m}$. The Picard sheaf $\mathrm{Pic}^\circ_\mathbb{Q}$ is semisimple and it is described in Lemma 1.1.

According to Lemma 1.1, for any simple abelian variety A over k, any non-zero element ξ of $\mathrm{Pic}^0(A)(k)_\mathbb{Q}$ provides an embedding of $\mathcal{H}_1^A$ into $\mathrm{Pic}^\circ_\mathbb{Q}$. Let us show that the natural extension $0 \to \mathcal{H}_1^{\mathbb{G}_m} \to \mathrm{Div}^\circ_\mathbb{Q} \to \mathrm{Pic}^\circ_\mathbb{Q} \to 0$ does not split, even after restricting to $\mathcal{H}_1^A$ via ξ.

All elements of $\mathrm{Pic}^\circ_\mathbb{Q}(A) := \mathrm{Pic}^0(A)_\mathbb{Q}$ are fixed by translations of A by torsion elements in $A(k)$. However, as the torsion subgroup in $A(k)$ is Zariski dense, it cannot fix a non-zero element of $\mathrm{Div}^\circ_\mathbb{Q}(A) := \mathrm{Div}_{\mathrm{alg}}(A)_\mathbb{Q}$.

This implies that $\mathcal{H}_1^{\mathbb{G}_m}$ is the maximal semisimple subsheaf of $\mathrm{Div}^\circ_\mathbb{Q}$, which proves, by Lemma 3.6, the second assertion. It follows also that the simple subquotients of V^1 are isomorphic to $\mathcal{H}_1^A$ for simple abelian k-varieties A. There are no extensions between $\mathcal{H}_1^A$ and $\mathcal{H}_1^B$ for abelian k-varieties A and B, since $\mathcal{I}_G$ is a Serre subcategory of $\mathcal{S}m_G$ by Proposition 2.2 and $Y \mapsto A(k(Y))_\mathbb{Q}$ is a projective object of $\mathcal{I}_G$ by property 5 of §1.2 and Proposition 2.10. This means that V^1 is semisimple, which proves the third assertion.

Once we know the simple subquotients of $\underline{\Omega}^1_{|k,\mathrm{closed}}$, the first assertion follows from Proposition 3.1 and Lemma 1.1. To see that the projections $\underline{\Omega}^1_{|k,\mathrm{closed}} \to \underline{\Omega}^1_{|k,\mathrm{closed}}/\Omega^1_{|k,\mathrm{reg}}$ and $\underline{H}^1_{\mathrm{dR}/k} \to \underline{H}^1_{\mathrm{dR}/k}/\underline{H}^1_{\mathrm{dR}/k,c}$ are split, it is enough to notice that V^1 is semisimple, and therefore, the compositions $\Omega^1_{|k,\mathrm{reg}} \hookrightarrow \underline{\Omega}^1_{|k,\mathrm{closed}} \to V^1$ and $\underline{H}^1_{\mathrm{dR}/k,c} \hookrightarrow \underline{H}^1_{\mathrm{dR}/k} \to V^1$ admit splittings. $\square$

Given a subfield L in F, define the filtration $N_\bullet^{(L)}$ on the $G_{F|L}$-modules W by $N_j^{(L)}W = \sum_{F'} W^{G_{F|F'}}$, where F' runs over the subfields in $F|L$ of transcendence degree j. (Clearly, $N_0^{(L)}W = W^{G_{F|\overline{L}}}$ and $N_\bullet^{(L)} = N_\bullet^{(\overline{L})} \subseteq N_\bullet^{(L')}$ if $L \subset L' \subset F$.) In particular, define the *level* filtration on the G-modules by $N_\bullet := N_\bullet^{(k)}$.

It is conjectured in [7, Conj.6.9] that the graded pieces of $N_\bullet$ on the objects of $\mathcal{I}_G$ are semisimple.

If U is an open subgroup of G, contained between $G_{F|\overline{L}}$ and the normalizer of $G_{F|\overline{L}}$ in G then $N_\bullet^{(L)}$ is a filtration by U-submodules. The forgetful functor $\mathcal{S}m_G \to \mathcal{S}m_U$ does not preserve the irreducibility (or the semisimplicity). E.g., for any commutative simple algebraic k-group A, the restriction to U of the irreducible G-module $A(F)/A(k)$ is a non-split extension of the irreducible U-module $A(F)/A(\overline{L})$ by the U- (in fact, $(U/U \cap G_{F|\overline{L}})$-) module $A(\overline{L})/A(k)$.

Questions. Let $W \in \mathcal{S}m_G$ be irreducible and $W = N_q W$. Is it true that the representation $W/N_{q-1}^{(L)}W$ of U is irreducible (or zero)?

Clearly, $N_j \Omega^i_{F|k} = \Omega^i_{F|k}$ for any $j > i$, $N_j \Omega^i_{F|k} = 0$ for any $j < i$ and $N_j \Omega^j_{F|k} \subseteq \Omega^j_{F|k,\mathrm{closed}}$.

Conjecture 3.8. $N_j \Omega^j_{F|k} = N_j \Omega^j_{F|k,\mathrm{closed}} = \Omega^j_{F|k,\mathrm{closed}}$.

A "weak" version, $N_j \Omega^j_{F|k,\mathrm{reg}} = \Omega^j_{F|k,\mathrm{reg}}$, follows from Grothendieck's diagonal decomposition conjecture.

The Conjecture obviously holds true for $j = 0$. The case $j = 1$ follows from (i) Proposition 3.7, (ii) the fact ([4, Cor.3.8]) that F/k and $F^\times/k^\times$ are acyclic, so $\Omega^1_{L|k,\mathrm{closed}} \twoheadrightarrow H^0(G_{F|L}, H^1_{\mathrm{dR}/k}(F)/kd\log(F^\times/k^\times))$, (iii) $N_1(A(F)/A(k)) = A(F)/A(k)$ for any commutative k-group A. $\square$

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